On the reach of Roberts' Theorem: A characterization of strategy-proof mechanisms independent of domain assumptions

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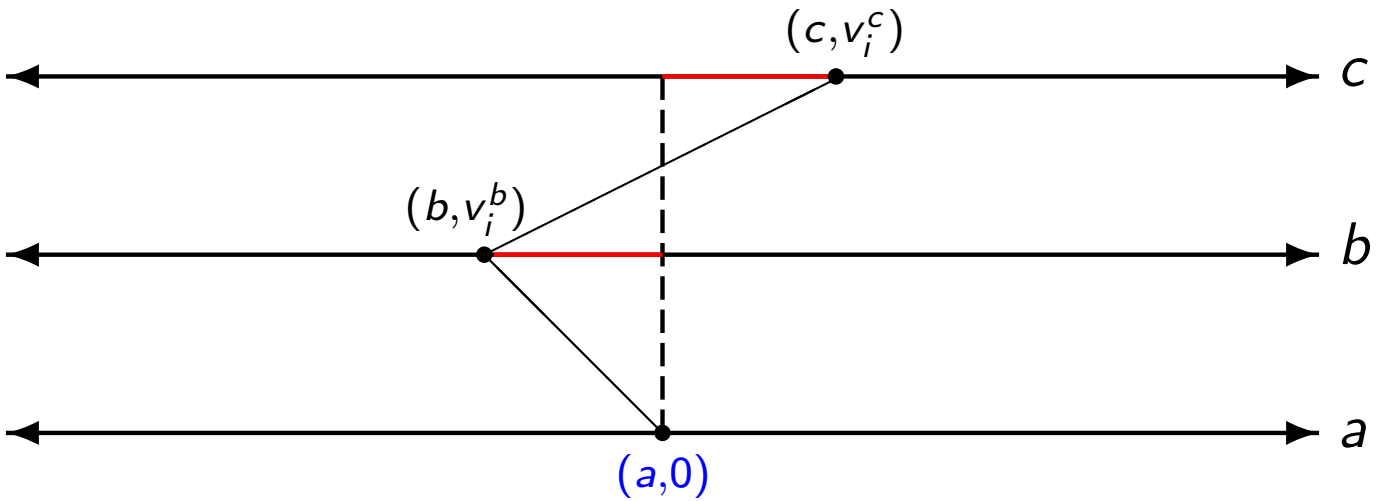
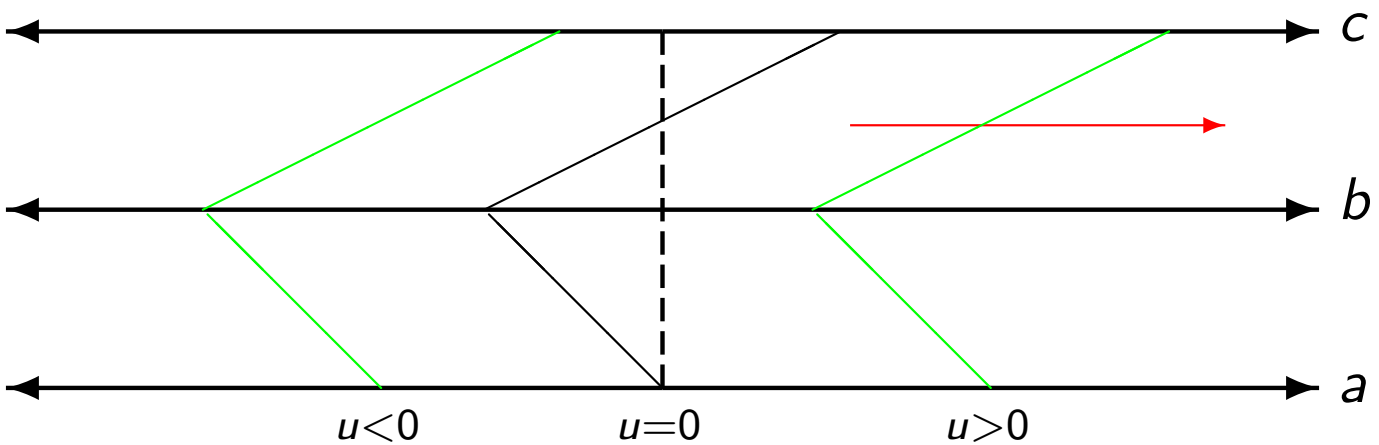
An Old Problem

- A finite set of alternatives $A = \{a, b, \ldots, z\}$
- A finite set of agents $I = \{1, 2, \ldots, n\}$
- For each agent $i \in I$, a *complete* and *transitive* preference relation $\succsim_i$ ranking the bundles in the 'consumption' set $A \times \mathbb{R}$
- For each agent $i \in I$, preferences are **quasi-linear**

Quasi-linear Preferences



Quasi-linear Preferences



The Utility Representation of $\succsim_i$

$$u_i\big((x,m);v_i\big) = \begin{cases} m - v_i^x & \text{if } x \in A \setminus \{a\}, \\ m & \text{otherwise.} \end{cases}$$

A solution concept

An **instance of the problem** is a profile of preferences

$$v_N = (v_1, v_2, \ldots, v_n) = \begin{bmatrix} v_1^b & v_1^c & \ldots & v_1^z \\ v_2^b & v_2^c & \ldots & v_2^z \\ \vdots & \vdots & & \vdots \\ v_n^b & v_n^c & \ldots & v_n^z \end{bmatrix} \in V \subseteq \mathbb{R}^{n \times (k-1)}.$$

A **solution** is an assignment of a bundle to each agent. Effectively, in doing so one needs to decide on an alternative $x \in A$ and a 'transfer' $(m_1, \ldots, m_n) \in \mathbb{R}^n$ which corresponds to the *allocation*

$$[(x, m_1), \ldots, (x, m_n)].$$

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What do we know

Roberts (1979): If $|A| \geq 3$ and for each $i \in N$, $V_i = \mathbb{R}^{n\times(k-1)}$(full domain), then $\varphi = (f, t)$ is Strategy-proof if and only if for each $v_N \in \mathbb{R}^{n\times(k-1)}$, $f(v_N)$ maximizes a weighted sum of the surplus (VCG territory).

$$f(v_N) \in \arg\max_{a\in A} \left\{ \sum_{i=1}^{n} w_i \, v_i(a) + c(a) \right\}.$$

Myesron (1981): If $|A| = 2$ (uni-dimensional case) there is a rich class of non-VCG mechanisms that are Strategy-proof

What we wish to better understand

Lavi et al (2009): "for all the intermediate range of domains, that are not single-dimensional nor unrestricted, almost no possibilities or impossibilities are known."

Mishra and Sen (2012): extend Roberts' Theorem for *k-dimensional open intervals* under Neutrality.

Carbajal et al. (2011): extend Roberts' Theorem by providing two sufficient conditions on the domain of valuations.

This Paper

A characterization of Strategy-proof mechanisms that does not rely on domain restrictions.

Strategy-proofness

A mechanism $\varphi = (f, m)$ *satisfies* Strategy-Proofness *if and only if for each* $v_N \in \mathbb{R}^{n\times(k-1)}$*, each* $i \in I$ *and each* $v_i' \in \mathbb{R}^{k-1}$

$$u\big(\varphi_i(v_i, v_{N\setminus\{i\}}); v_i\big) \geq u\big(v_i(v_i', v_{N\setminus\{i\}}); v_i\big).$$

Proposition

A rule satisfies *Strategy-proofness* for each $v_N \in V_i$ and each $i \in I$ either $f(v_i, v_{-i}) = a$ for som $a \in A$ and each $v_i \in V_i$, or for each $x \in A$ there exists $v_i^x \in V_i$ such that $f(v_i^x, v_{-i}) = x$.

Agent $i \in I$ is either fully *soveriegn* or completely irrelevant at each profile $v_{-i} \in V_{-i}$

The principal finding (Full Domain)

Theorem

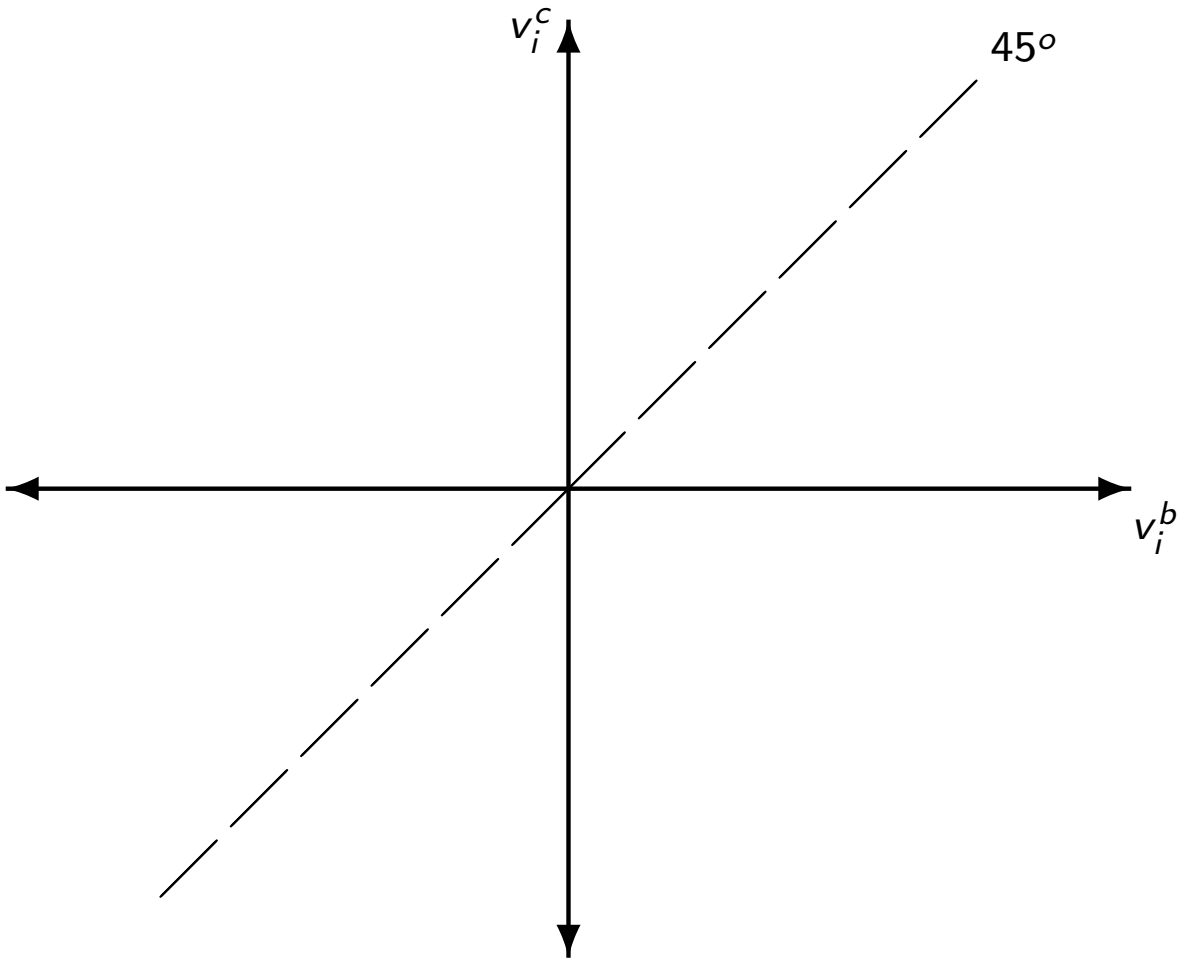
A rule satisfies Strategy-proofness and Responsiveness if and only if there exist n pairs of functions $p_i : \mathbb{R}^{(n-1)\times(k-1)} \to \mathbb{R}$ *(non-increasing) and* $h_i : \mathbb{R}^{(n-1)\times(k-1)} \to \mathbb{R}$ *such that for each* $v_N \in V_i$ *and each* $i \in I$

$$f(v_N) \in \begin{cases} \{a\} & \text{if } v_{N\setminus\{i\}} \gg p_i(v_{N\setminus\{i\}}), \\ \{x \in A \,:\, v_i^x \leq v_i^y,\ \forall y \in A\} & \text{if } \neg v_{N\setminus\{i\}} \geq p_i(v_{N\setminus\{i\}}), \\ \{x \in A \,:\, v_i^x \leq v_i^y,\ \forall y \in A\} \cap \{a\} & \text{otherwise,} \end{cases}$$

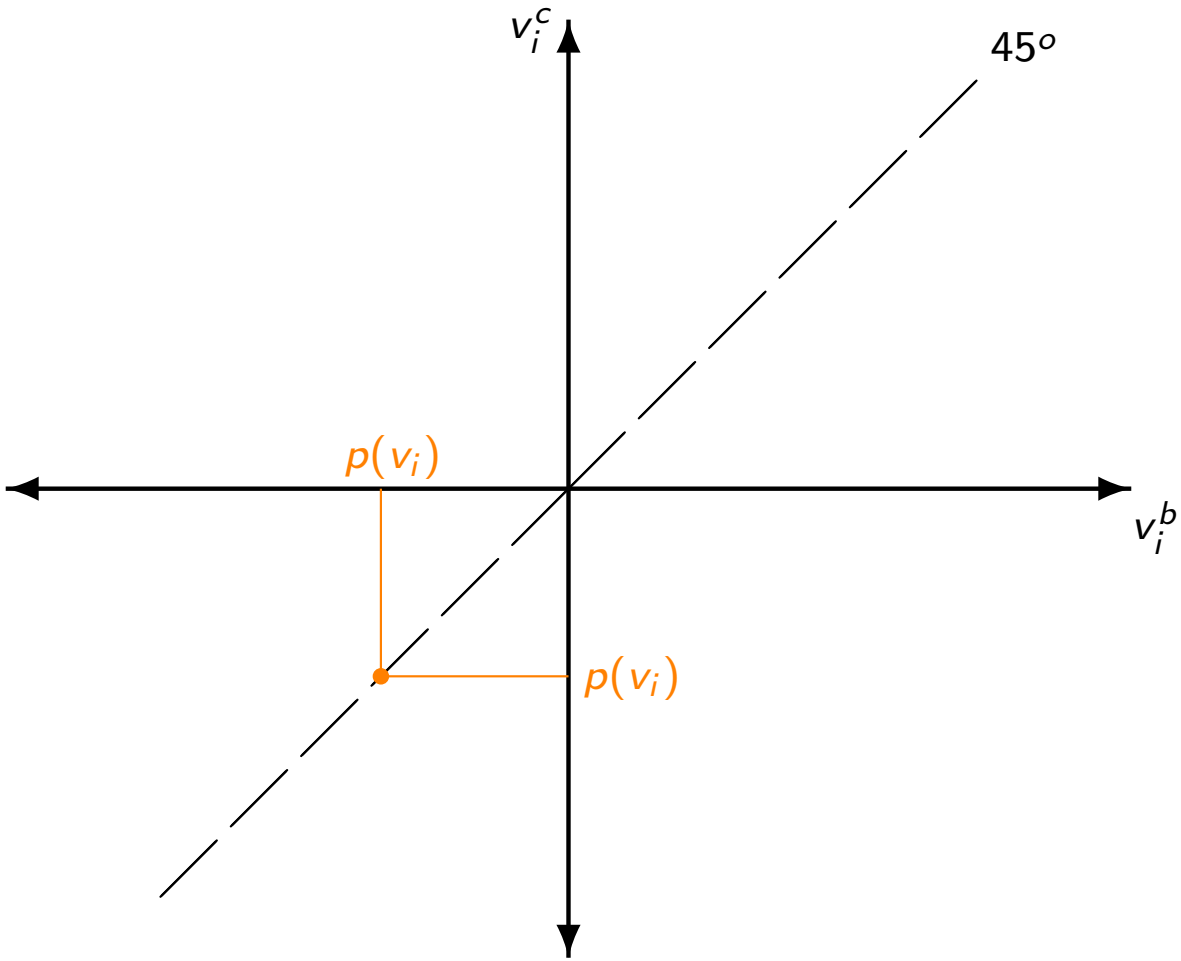
and

$$m_i(v_N) \in \begin{cases} -p_i(v_{N\setminus\{i\}}) + h_i(v_{N\setminus\{i\}}) & \text{if } f(v_N) = a, \\ h_i(v_{N\setminus\{i\}}) & \text{otherwise.} \end{cases}$$

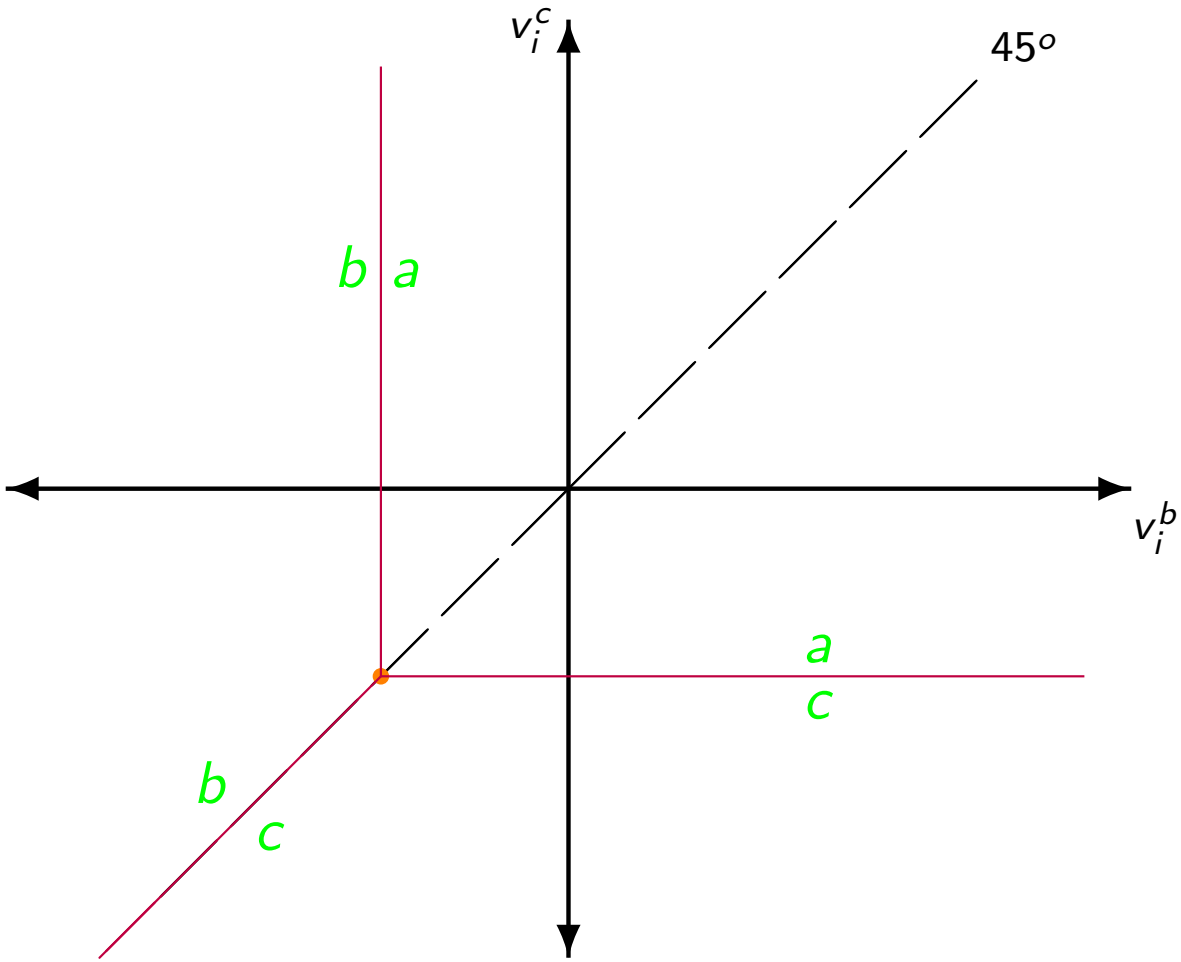
Fix v_{-i} and consider V_i



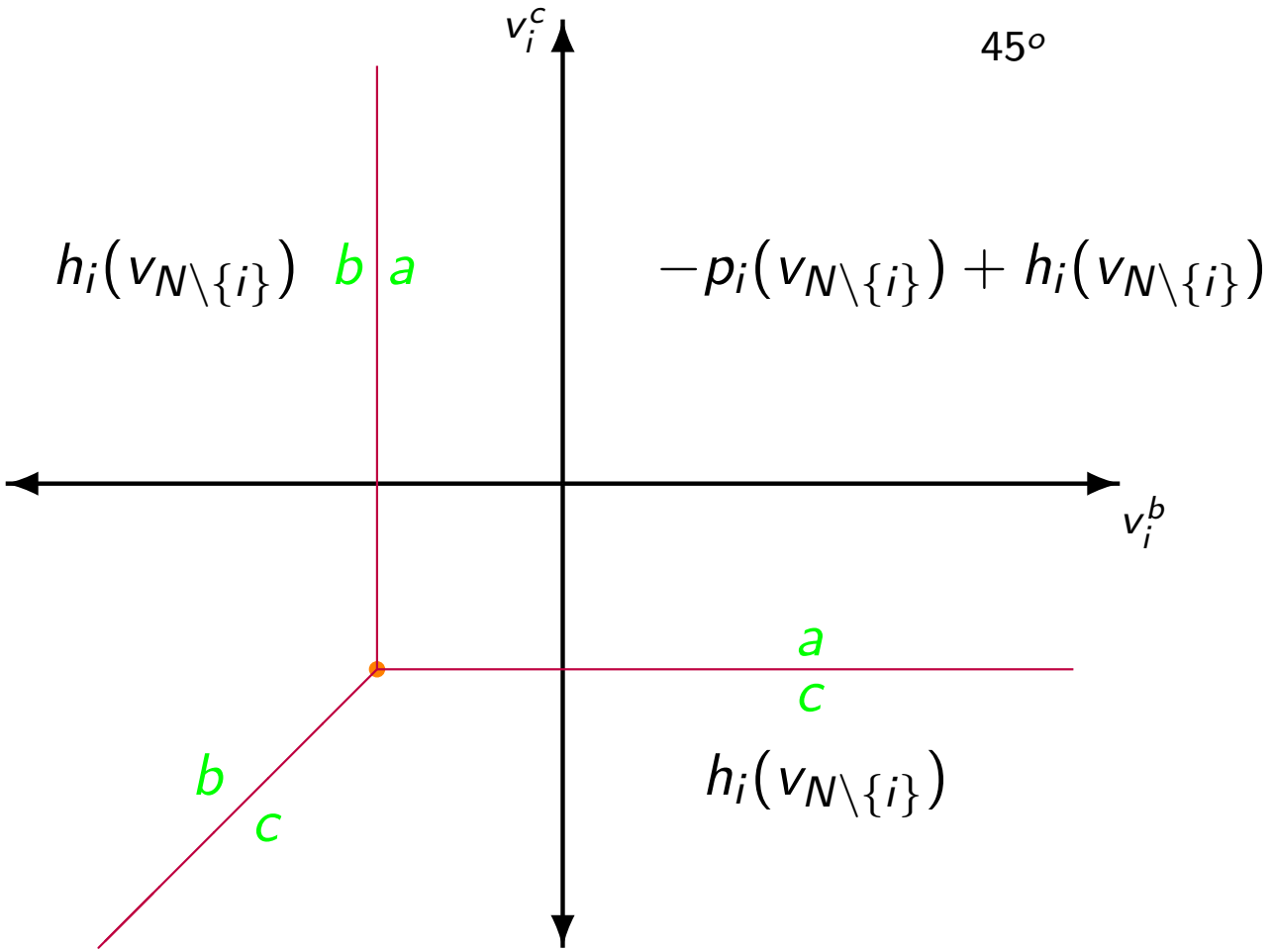
$p(v_{N\setminus\{i\}})$ selects a point on the 45^o line



The parameter $p_i(v_{-i})$ produces a partition*



The parameter $p_i(v_{N\setminus\{i\}})$ influences transfers



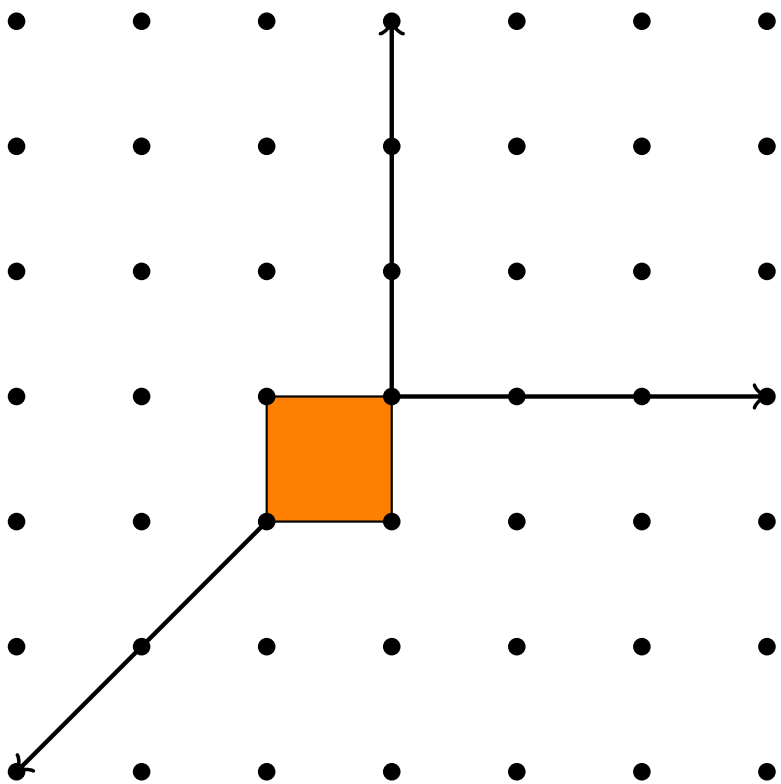
General Domain: $V_i \subset \mathbb{R}^{k-1}$

For each $x \in A \setminus \{a\}$, the function $m^x(v_{-i})$ (the payment scheme if alternative a is chosen) takes a values in the interval

$$\left[\sup\{q \in V_i^x : q < p^x(v_{-i})\}, p^x(v_{-i})\right]$$

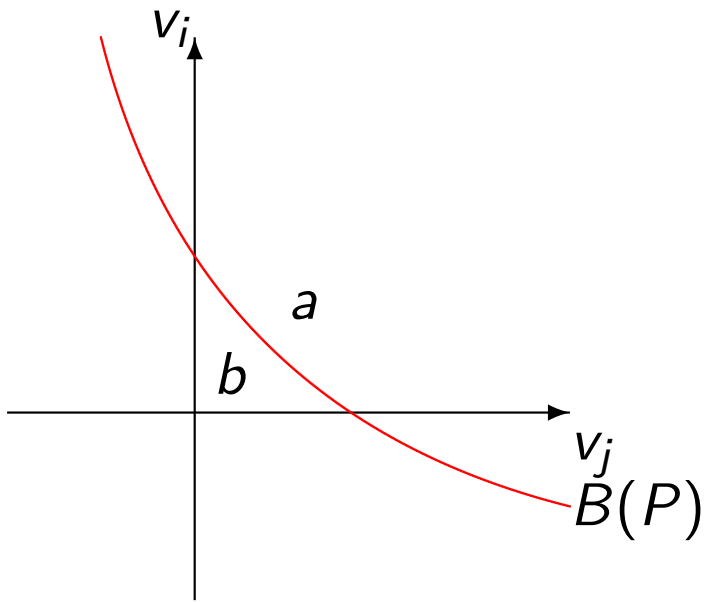
where $\left(p^x(v_{-i})\right)_{x \in A\setminus\{a\}} = \inf_{>}\left\{v_i \in \mathbb{R}^{k-1} \ : \ f\left(v_i, v_{-i}\right) = a\right\}$

Example: $V_i = \mathbb{N}^2$



The Uni-dimensional case (Nisan 2007)

Let $\mathcal{P}$ be the set of functions $P : \mathbb{R}^n \to \{a, b\}$ that are weakly monotonic

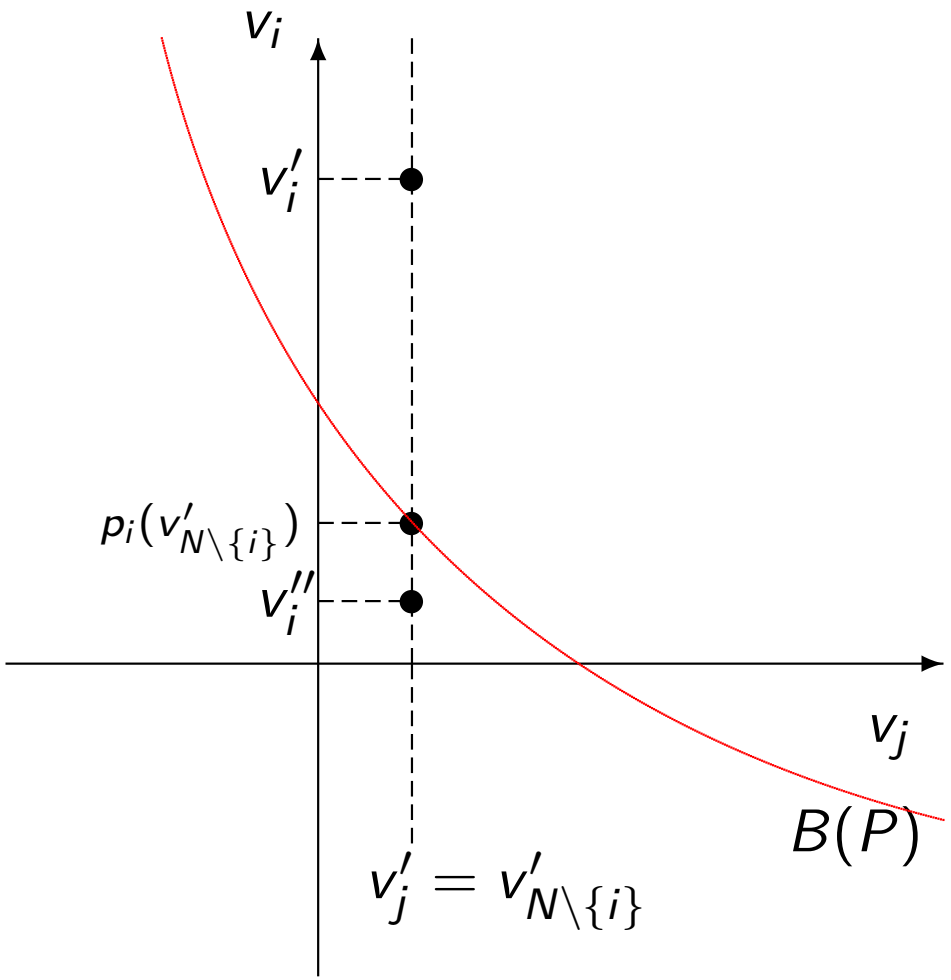


A mechanism $\varphi = (f, t)$ is strategy-proof only if there exists $P \in \mathcal{P}$ such that

$$\{\theta_N \in \mathbb{R}^n \ : \ P(\theta_N) = 1\} = \{\theta_N \in \mathbb{R}^n \ : \ f(\theta_N) = a\},$$
$$\{\theta_N \in \mathbb{R}^n \ : \ P(\theta_N) = 0\} = \{\theta_N \in \mathbb{R}^n \ : \ f(\theta_N) = b\}.$$

Let $B(P)$ denote the **boundary** of such partition

A representation Theorem



A representation Theorem

For each $P \in \mathcal{P}$, each $\theta_N \in \mathbb{R}^n$ and each $i \in N$, let

$$p_i(\theta_{N\setminus\{i\}}) = \begin{cases} \inf\{x \in \mathbb{R} : P(x; \theta_{N\setminus\{i\}}) = 1\} & \text{if } P(\mathbb{R}, \theta_{N\setminus\{i\}}) = \{0,1\}, \\ 0 & \text{if } P(\mathbb{R}, \theta_{N\setminus\{i\}}) = \{k\} \\ & \text{for some } k \in \{0,1\}. \end{cases}$$

A representation Theorem (Nisan, 2007)

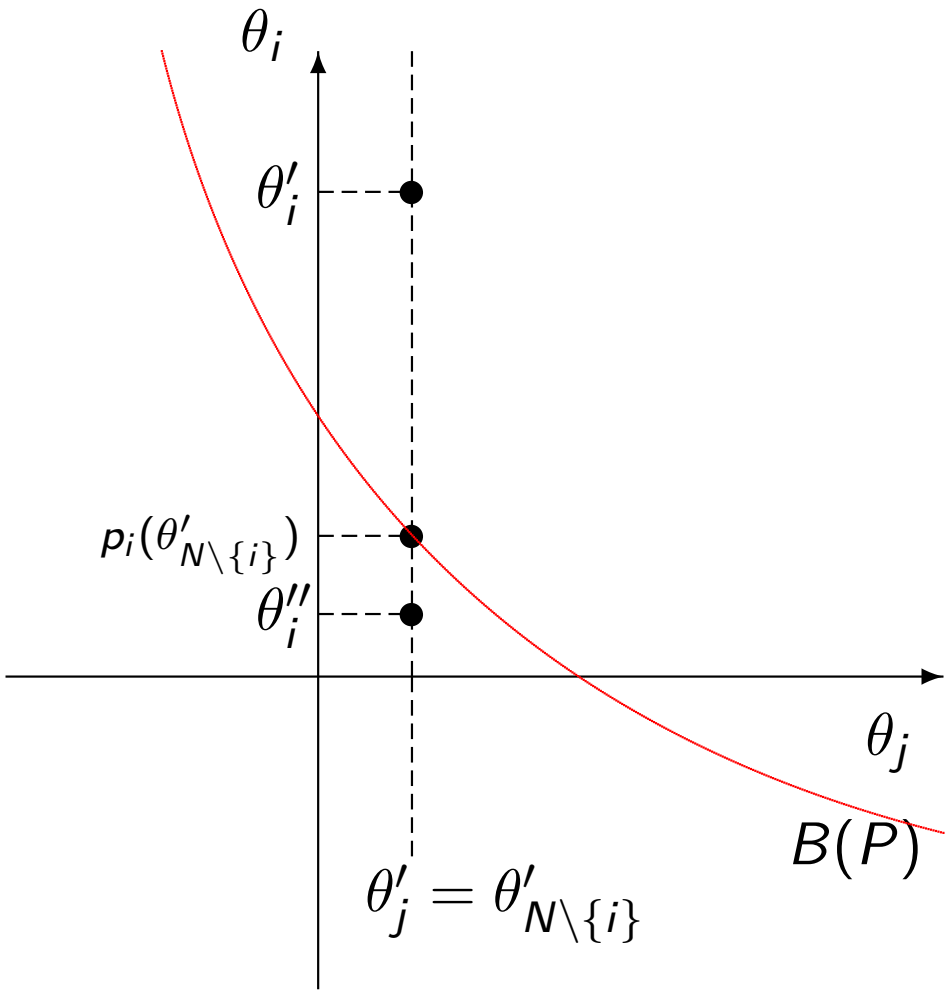
Let $\mathcal{G}$ be the set of functions $g_i : \mathbb{R}^{n-1} \to \mathbb{R}$.

A mechanism $\varphi = (f, t)$ *is strategy-proof if and only if there exist* $P \in \mathcal{P}$ and $(g_i)_{i \in N} \in \mathcal{G}^n$ *such that for all* $\theta_N \in \mathbb{R}^n$,

$$\varphi(\theta_N) = \begin{cases} \left(1, \left(g_i(\theta_{N\setminus\{i\}}) - p_i(\theta_{N\setminus\{i\}})\right)_{i\in N}\right) & \text{if } P(\theta_N) = 1 \\ \left(0, \left(g_i(\theta_{N\setminus\{i\}})\right)_{i\in N}\right) & \text{if } P(\theta_N) = 0 \end{cases}$$

Every strategy-proof mechanism can be associated with a **payment scheme** $(P, (g_i)_{i \in N})$.

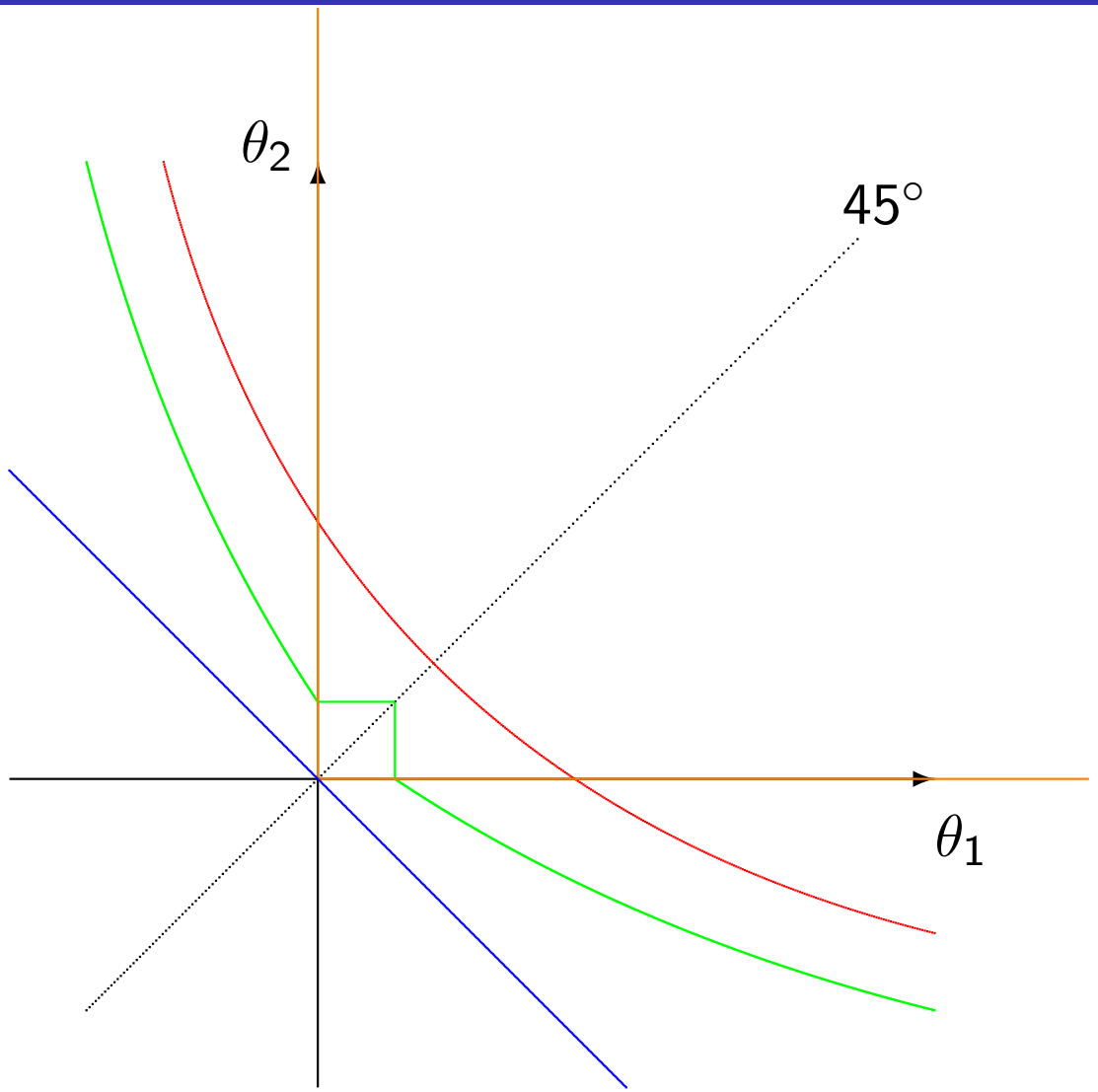
A representation Theorem



$$t_i\left(\theta''_i, \theta'_{N\setminus\{i\}}\right) = g_i(\theta'_{N\setminus\{i\}})$$

$$t_i\left(\theta'_i, \theta'_{N\setminus\{i\}}\right) = -p_i(\theta'_{N\setminus\{i\}}) + g_i(\theta'_{N\setminus\{i\}})$$

Examples



Concluding Remarks

- A project that aims at understanding the nature of Strategy-proofness.
- Two key questions remain open:
  1. Flesh out the implications of the characterization: Can it be as useful generally, as it is in the uni-dimensional case?
  2. Can we use it to identify specific domain restrictions that enable implementation possibilities beyond affine maximizers?

Decision Efficiency

A mechanism $\varphi = (f, t)$ *satisfies* Decision Efficiency *if and only if for each* $\theta_N \in \mathbb{R}^n$

$$\sum_{i \in N} \theta_i > 0 \implies f(\theta_N) = 0,$$

$$\sum_{i \in N} \theta_i < 0 \implies f(\theta_N) = 1.$$

Vickrey-Clarke-Groves

A simple application of the representation Theorem...

A *Strategy-Proof* mechanism $\varphi = (f, t)$, associated with payment scheme (P, g) satisfies *Decision Efficiency* if and only if for each $\theta_N \in \mathbb{R}^n$ and each $i \in N$, $p(\theta_{N\setminus\{i\}}) = -\sum_{j\in N\setminus\{i\}} \theta_j$.

The properties of the mechanism(s)... one piece at a time

2) **No money injected from the "outside"...**

A mechanism $\varphi = (f, t)$ satisfies **Feasibility** if and only if for each $\theta_N \in \mathbb{R}^n$,

$$\sum_{i \in N} t_i(\theta_N) \leq 0.$$

The properties of the mechanism(s)... one piece at a time

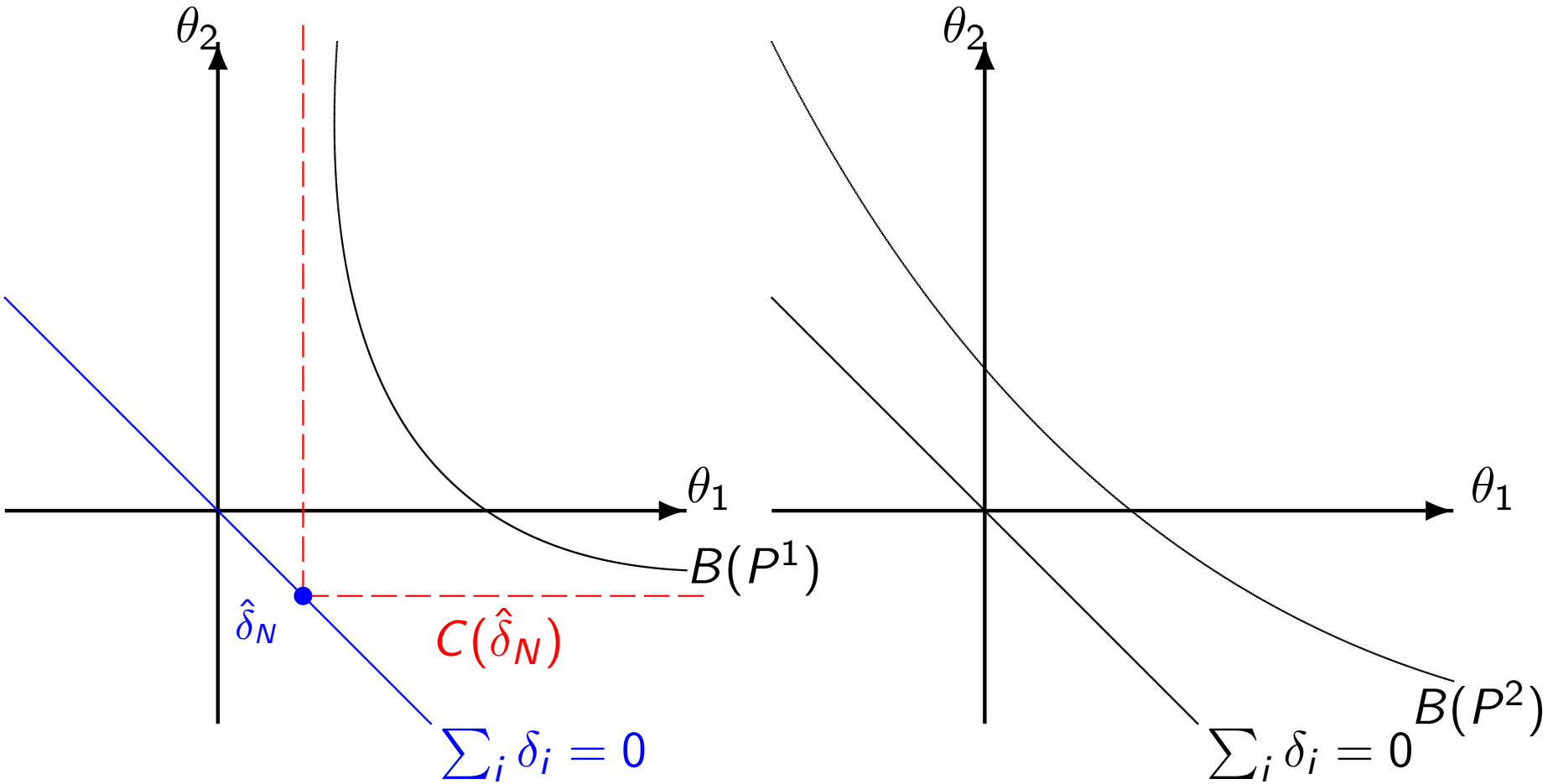
3) **No one should be coerced into participation...** (every agent should be at least as well as at the status quo)

A mechanism $\varphi = (f, t)$ satisfies **Voluntary Participation** if and only if for each $\theta_N \in \mathbb{R}^n$ and each $i \in N$, $u\big(\varphi_i(\theta_N); \theta_i\big) \geq 0$.

A baseline result...

Let $\Delta_N = \left\{\delta_N \in \mathbb{R}^n \ : \ \sum_{i \in N} \delta_i = 0\right\}$; for each $\delta_N \in \Delta_N$ let

$$C(\delta_N) = \left\{\theta_N \in \mathbb{R}^n \ : \ \forall i \in N, \ \theta_i \geq \delta_i\right\}.$$



A baseline result...

Theorem

A mechanism $\varphi = (f, t)$*, associated with the payment scheme*

$$\left(P, (g_i)_{i \in N}\right) \in \mathcal{P} \times \mathcal{G}^n,$$

satisfies Strategy-proofness, Feasibility and Voluntary Participation if and only if $P \in \mathcal{P}$ *is such that either* $P(\mathbb{R}^n) = \{0\}$*, or there exists* $\delta_N \in \Delta_N$ *such that,*

$$\left\{\theta_N \in \mathbb{R}^n \ : \ P(\theta_N) = 1\right\} \subseteq C(\delta_N),$$

and for each $i \in N$*,* $g_i = g^0 : \mathbb{R}^{n-1} \to \{0\}$*.*

The properties of the mechanism(s)... one piece at a time

4) **No mechanism should be dominated by another mechanism**... A

mechanism φ' **dominates** φ, if and only if,

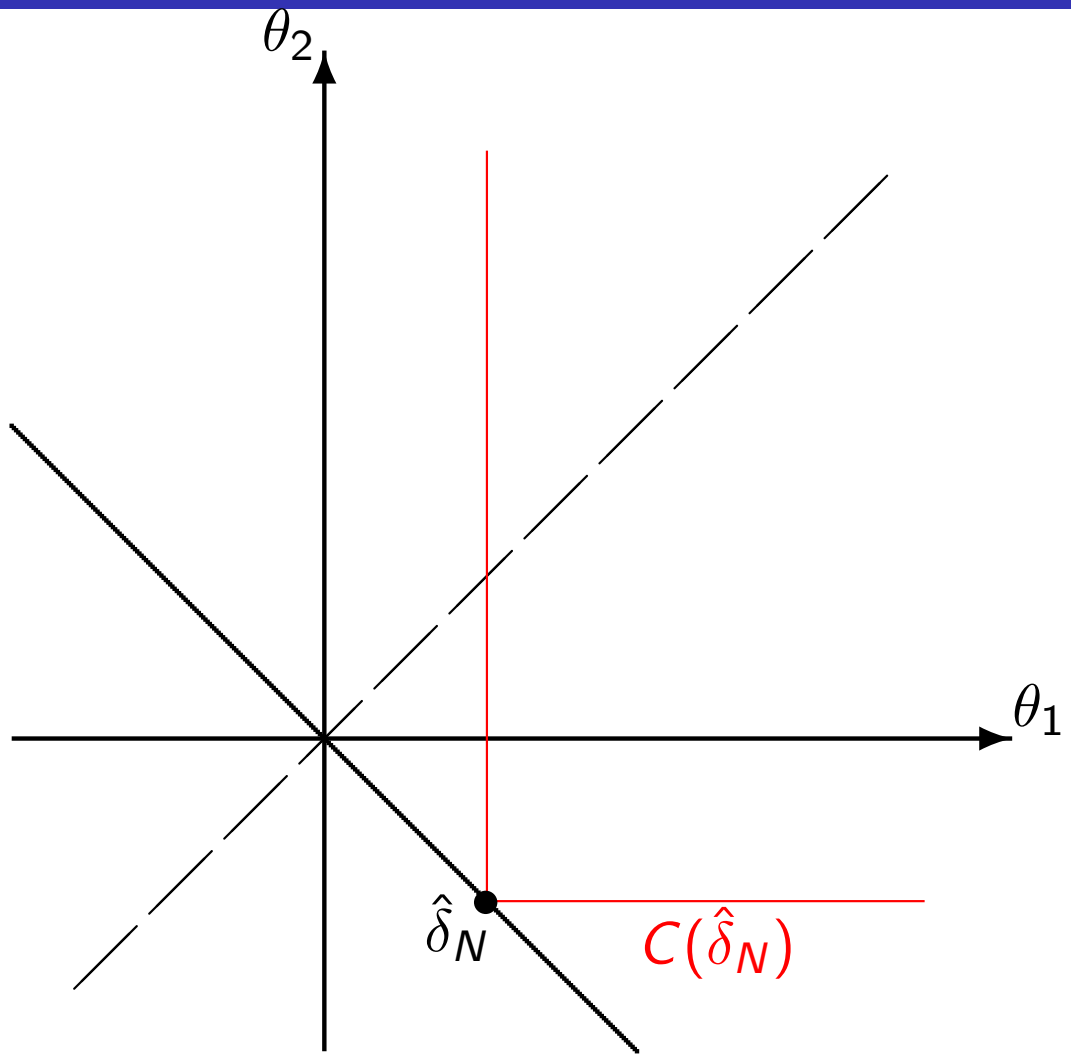
1. for each $\theta_N \in \mathbb{R}^n$ and each $i \in N$, $u\big(\varphi'_i(\theta_N); \theta_i\big) \geq u\big(\varphi_i(\theta_N); \theta_i\big)$,
2. for some $\tilde{\theta}_N \in \mathbb{R}^n$ and some $j \in N$, $u\big(\varphi'_j(\tilde{\theta}_N); \tilde{\theta}_j\big) > u\big(\varphi_j(\tilde{\theta}_N); \tilde{\theta}_j\big)$.

A mechanism φ is **Undominated** if and only if it is *feasible* and *strategy-proof* and, moreover, there does not exist another *feasible*, *anonymous* and *strategy-proof* mechanism φ' that dominates φ.

Question

What remains of the family we established before?

Reference-consensus mechanism



$$\left\{\theta_N \in \mathbb{R}^n \ : \ P(\theta_N) = 1\right\} = C(\hat{\delta}_N)$$

For each $i \in N$, $g_i = g^0$

Our principal finding

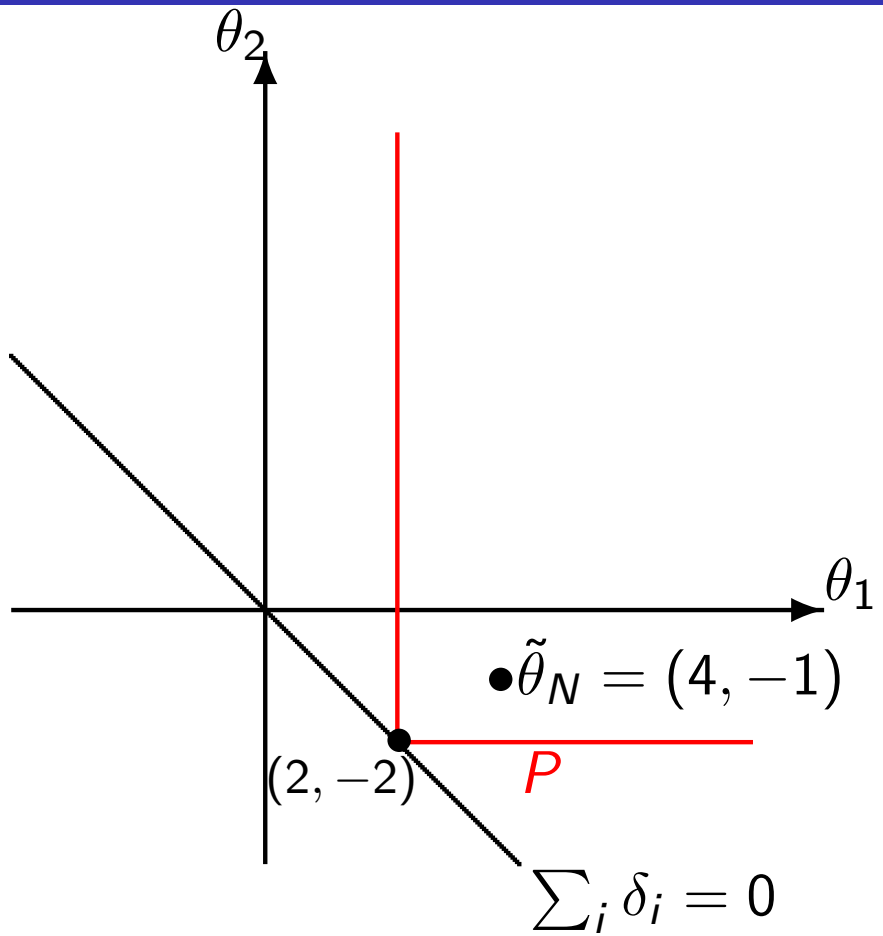
Theorem

A mechanism $\varphi = (f, t)$*, associated with the payment scheme*

$$\left(P, (g_i)_{i \in N}\right) \in \mathcal{P} \times \mathcal{G}^n,$$

satisfies Strategy-proofness, Feasibility, Voluntary Participation and Undominatedness if and only if it is a reference-consensus mechanism.

Example



$$P(\tilde{\theta}_N) = 1$$

$$t_1 = -p_1(\tilde{\theta}_2) = -2; U_1 = 4 - 2 = 2$$

$$t_2 = -p_2(\tilde{\theta}_1) = 2; U_2 = -1 + 2 = 1$$

Some issues...

The mechanism in the previous example

- Is biased in favor of agent 2 (always subsidized)
- Is biased in favor of the status-quo
- The only reference vector avoiding both these occurrences is the vector $(0, \ldots, 0) \in \Delta_N$

The properties of the mechanism(s)... one piece at a time

A mechanism $\varphi = (f, t)$ *satisfies* **Anonymity** *if and only if for each* $\theta_N \in \mathbb{R}^n$, *each* $\pi \in \Pi$ *and each* $i \in N$,

$$\varphi_i(\theta_N) = \varphi_{\pi(i)}\big(\pi(\theta_N)\big).$$

A mechanism $\varphi = (f, t)$ *satisfies* **Weak Neutrality** *if and only if for each* $y \in \mathbb{R} \setminus \{0\}$,

$$f(y, \ldots, y) \neq f(-y, \ldots, -y).$$

A Corollary...

Theorem

A mechanism φ satisfies Strategy-proofness, Feasibility, Voluntary Participation, Undominatedness and Anonymity if and only if it is the Unanimity mechanism. Similarly, a mechanism φ satisfies Strategy-proofness, Feasibility, Voluntary Participation, Undominatedness and Weak neutrality if and only if it is the Unanimity mechanism.

Discussion

- In our setting VP becomes compelling because a status-quo exists (there is eventually a threshold to be met...)
- VP is not so restrictive after all
- Undominatedness constitutes a fundamental element of the normative argument for Unanimity. As much so, as Voluntary Participation.

What if there is not a status quo?

There are *too many* Undominated Mechanisms...
...One needs to restrict such a big set of options...

Weakening Assignment efficiency

A mechanism $\varphi = (f, t)$ *satisfies* **Weak Neutrality** *if and only if for each* $y \in \mathbb{R} \setminus \{0\}$,

$$f(y, \ldots, y) \neq f(-y, \ldots, -y).$$

A mechanism $\varphi = (f, t)$ *satisfies* **Decision Efficiency** *if and only if for each* $\theta_N \in \mathbb{R}^n$

$$\sum_{i \in N} \theta_i > 0 \implies f(\theta_N) = 0,$$

$$\sum_{i \in N} \theta_i < 0 \implies f(\theta_N) = 1.$$

$DE \implies WN$

Remark: Weak Neutrality on its own isolates those $P \in \mathcal{P}$ satisfying $(0, \ldots, 0) \in B(P)$...

What if there is not a status quo?

A mechanism $\varphi = (f, t)$ *satisfies* **No Positive Transfers to the Winners** *if and only if for each* $\theta_N \in \mathbb{R}^n$ *and each* $i \in N$,

$$f(\theta_N) = 1 \text{ and } \theta_i > 0 \implies t_i(\theta_N) \leq 0, \text{ and}$$

$$f(\theta_N) = 0 \text{ and } \theta_i < 0 \implies t_i(\theta_N) \leq 0.$$

A mechanism $\varphi = (f, t)$ *satisfies* **No-Envy** *if and only if for each* $\theta_N \in \mathbb{R}^n$ *and each* $i, j \in N$, $u(\varphi_i(\theta_N); \theta_i) \geq u(\varphi_j(\theta_N); \theta_i)$.

$NE \implies NPTtW$

Theorem 1 (Athanasiou and Valletta 2021)

Suppose that $\varphi = (f, t)$, associated with the payment scheme $(P, g) \in \mathcal{P} \times \mathcal{G}$, is a strategy-proof, anonymous, feasible and undominated mechanism. Mechanism φ satisfies Weak Neutrality and No Positive Transfers to the Winners if and only if it satisfies Conditions W and S, $(0, \ldots, 0) \in B(P)$, and for each $\theta_N \in \mathbb{R}^n$ and each $i \in N$,

$$g\left(\theta_{N\setminus\{i\}}\right) = \min\left\{0, p(\theta_{N\setminus\{i\}})\right\}.$$

Once one has chosen one of the many decision criteria complying with Theorem 1... then the transfers are **uniquely** determined:

$$t_i = -p\left(\theta_{N\setminus\{i\}}\right) + g\left(\theta_{N\setminus\{i\}}\right) = -p\left(\theta_{N\setminus\{i\}}\right) + \min\left\{0, p(\theta_{N\setminus\{i\}})\right\}.$$

Corollaries

Corollary 1 of Theorem 1 *The Pivotal mechanism is the only undominated Groves mechanism satisfying No Positive Transfers to the Winners.*

Corollaries

A mechanism $\varphi = (f, t)$ *satisfies* **Budget-balance** *if and only if for each* $\theta_N \in \mathbb{R}^n$, $\sum_{i \in N} t_i(\theta_N) = 0$.

Corollary 2 of Theorem 1 *A mechanism belonging to the class characterized by Theorem 1 satisfies Budget-Balance if and only if it satisfies No-Envy.*

Voting Rules

For each $\theta_N \in \mathbb{R}^n$, let

$a(\theta_N) = \left|\{i \in N \ : \ \theta_i > 0\}\right|$ and $b(\theta_N) = \left|\{i \in N \ : \ \theta_i < 0\}\right|$.

A family of (strongly decisive) special majority voting rules (Fishburn 1973) that rely solely on the information conveyed by the paprameters above.

Corollaries

Corollary 3 of Theorem 1 *Suppose that $\varphi = (f, t)$, associated with the payment scheme $(P, g) \in \mathcal{P} \times \mathcal{G}$, is a strategy-proof, anonymous, feasible and undominated mechanism. Mechanism φ satisfies Weak Neutrality and No Envy if and only if there exists a finite sequence of integers $s(0), s(1), ..., s(m-1)$ with $m \leq n$ such that:*

i. $s(k) \in \{0, 1\}$ *if* $k = 0$ *and* $s(k) \in \{1, n-k\}$ *if* $k = 1, ..., m-1$;

ii. $s(0) \leq s(1) \leq \ ... \ \leq s(m-1)$;

iii. *for each* $\theta_N \in \mathbb{R}^n$, $P(\theta_N) = 1$ *if and only if*

$$b(\theta_N) < m \text{ and } a(\theta_N) \geq s\big(b(\theta_N)\big);$$

iv. *for each* $\theta_N \in \mathbb{R}^n$ *and each* $i \in N$, $t_i(\theta_N) = 0$.

For instance

Let N=$\{1,2,3\}$

$m = 1$ and $s(0) = 1$; if just one agent prefers b to a then b is chosen

$m = 3$ and $s(0) = 0$; b is chosen only if three agents prefer b to a

$m = 2$, $s(0) = 1$ and $s(1) = 1$; a is chosen if $1 \leq b(\theta_N) < 2$

So far...

- A proof of concept for *Undominatedness*: A workable concept with meaningful consequences that is applicable to mechanism design in general.
- A new family of mechanisms that places the focus on the many different ways the issue of tension between decision efficiency and budget-balance may be resolved.

  An embarrassment of riches: Does some mechanism emerge as a definitive proposal?