How Small is Small?

Non-linearities in Heterogeneous Agent Models

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- BUT: How small is small?
- Our answer: for fiscal transfers, "small" must be *very* small.

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- Local methods are unreliable when response to shocks are driven by failures of Ricardian equivalence
  - Shocks where they *are* accurate are those where HA $\approx$ RA.

Roadmap

1. Partial equilibrium model and calibration
2. Fiscal stimulus in general equilibrium
3. Other shocks

Environment: Household Problem

- Continuum of households with idiosyncratic income shocks

$$\max_{\{c_{i,t},a_{i,t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$

$$\text{s.t.} \quad c_{i,t} + a_{i,t+1} \leq (1+r)a_{i,t} + (1-\tau)y_{i,t} + T_t \quad \text{and} \quad a_{i,t+1} \geq 0.$$

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Intertemporal Euler equation:

$$u'(c_{i,t}) \geq \beta_i(1+r)\mathbb{E}_t u'(c_{i,t+1}) \quad \text{with equality if } a_{i,t+1} > 0.$$

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Given initial distribution Γ_0 and $c_t(x,y)$, where $x = (1+r)a + (1-\tau)y + T$, aggregate consumption function:

$$\mathcal{C}_t(\{T_s\}) = \int c_t(x,y) d\Gamma_t(x,y)$$

The Policy

Starting from steady state, a one-time unanticipated lump-sum transfer at $t = 0$:

$$T_0 = T + \Delta, \qquad T_t = T \quad \text{for } t > 0.$$

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Object of interest. Aggregate consumption response:

$$\mathcal{C}_t(\Delta) - \mathcal{C}_t(0)$$

Next: linear approximation vs. the exact non-linear solution.

Model Computation

Non-linear solution

- Use household policy (solved exactly) and aggregate

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Linear solution

- Taylor expansion of aggregate consumption function around steady state

$$\mathcal{C}_t(\Delta) = \mathcal{C}_t(0) + \left.\frac{\partial \mathcal{C}_t}{\partial \Delta}\right|_{\Delta=0} \cdot \Delta$$

Slope computed analytically, by finite differences, or with SSJ

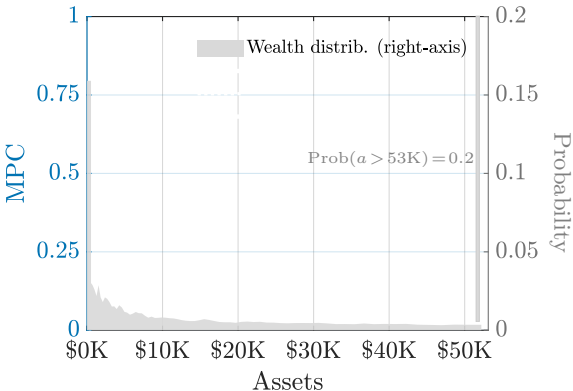
Calibration

- A period = a quarter.
- Earnings process, $y_{i,t}$, subject to temporary and persistent shocks
  - Calibrated as in Kaplan-Violante (2022) to match second moments of households' annual earnings
- Interest rate 2% annual.
- Tax rate (τ) = 27.4%
- Steady-state transfers = 15% of GDP ($\approx$ \$25K/yr)
- Heterogenous discount factors:

$$\left.\begin{array}{l}\text{Mean wealth / annual GDP} = 100\% \ (\approx \$166\text{K}) \\ \text{Avg. quarterly MPC (out of \$1,000)} = 25\%\end{array}\right\} \Rightarrow \ \bar{\beta} = 0.972, \ \sigma_\beta = 0.010$$

Steady-state wealth distribution and MPCs

(a) Wealth distribution



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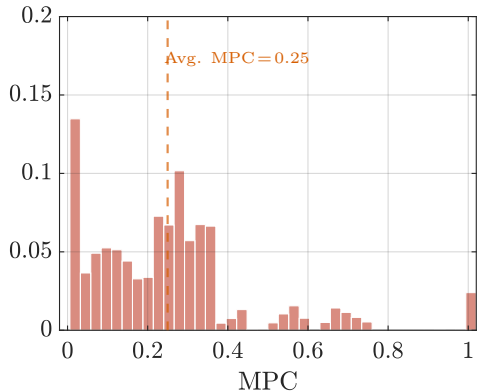

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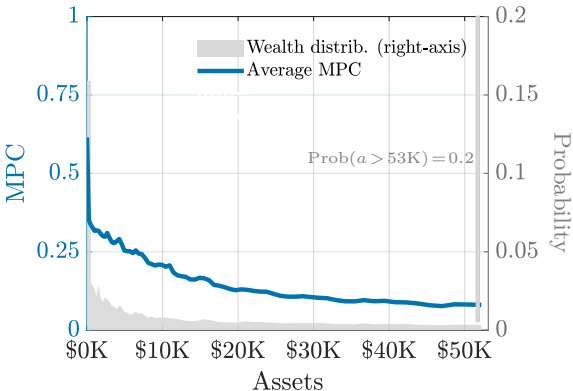

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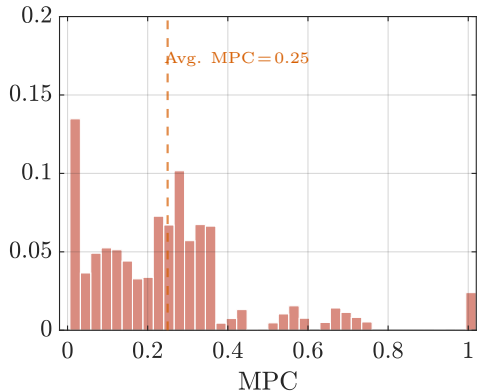

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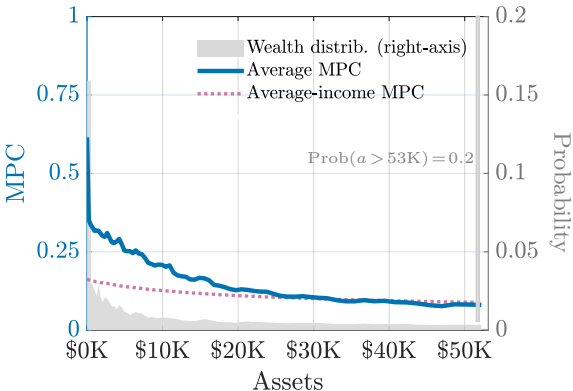

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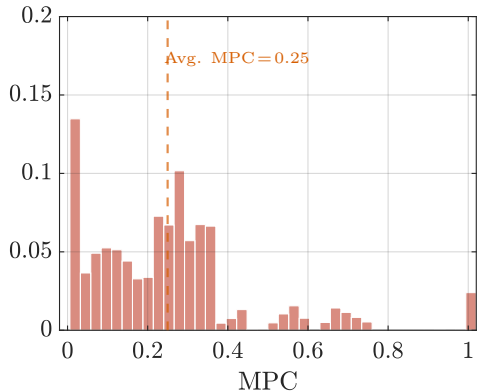

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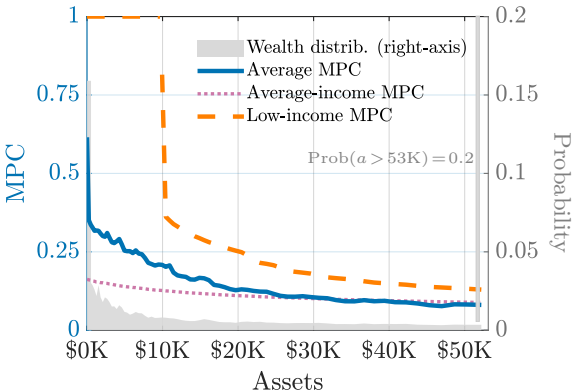

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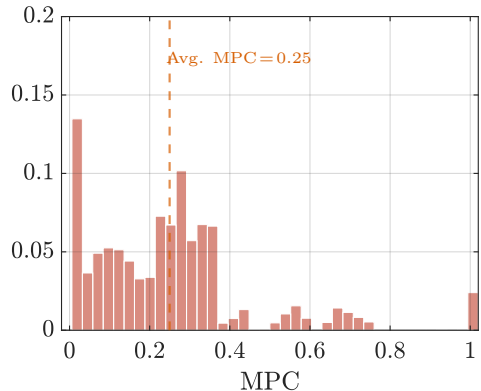
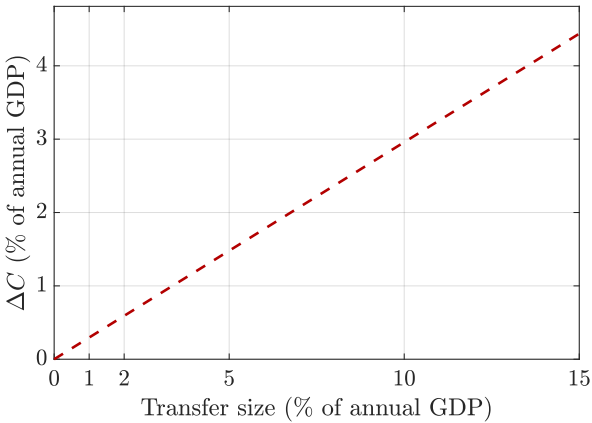


Aggregate consumption response



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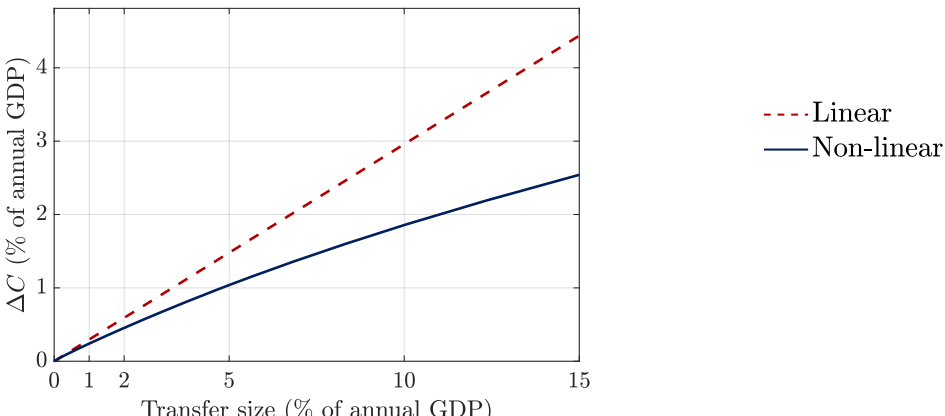

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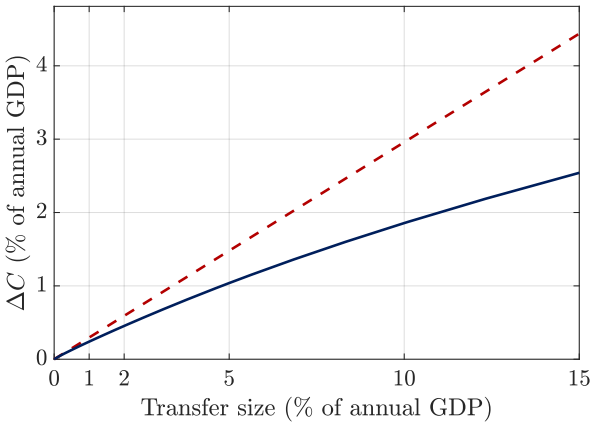

- Non-liner solution is concave
- Linear overstates response:
  - At $\Delta = 5\%$ of GDP ($\approx \$8K$): overstates by 42%
  - At $\Delta = 0.5\%$: overstates by 17%

Understanding the Gap: Two sources of Concavity

- Concavity of individual consumption function, as in Carroll-Kimball
- Kink from the borrowing constraint

Understanding the Gap: Two sources of Concavity

$$\mathcal{C}(\Delta) = \int_Y \Big[\int_{-\infty}^{\bar{x}(y)-\Delta} c^{\text{con}}(x+\Delta, y)\, \mathrm{d}G(x \mid y) \;+\; \int_{\bar{x}(y)-\Delta}^{\infty} c^{\text{unc}}(x+\Delta, y)\, \mathrm{d}G(x \mid y) \Big] \mathrm{d}F(y)$$

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$$\mathcal{C}'(0) = \int_Y G(\bar{x}(y) \mid y)\, \mathrm{d}F(y) + \int_Y \int_{\bar{x}(y)}^{\infty} c_x(x, y)\, \mathrm{d}G(x \mid y)\, \mathrm{d}F(y)$$

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MPC below 1, falls with wealth

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Linear solution simply extrapolates average MPC

- Changes in constrained share irrelevant to first-order (by continuity of c)
  - As if constrained households MPC=1 permanently

Concavity: second order

$$\mathcal{C}''(0) = \underbrace{-\int_Y \overbrace{(1 - c_x(\bar{x}(y), y))}^{\text{difference in MPC at the boundary}} \overbrace{g(\bar{x}(y) \mid y)\, \mathrm{d}F(y)}^{\text{probability of crossing to unconstrained}}}_{\text{extensive}} + \underbrace{\int_Y \int_{\bar{x}(y)}^{\infty} c_{xx}(x, y)\, \mathrm{d}G(x \mid y)\, \mathrm{d}F(y)}_{\text{intensive}}$$

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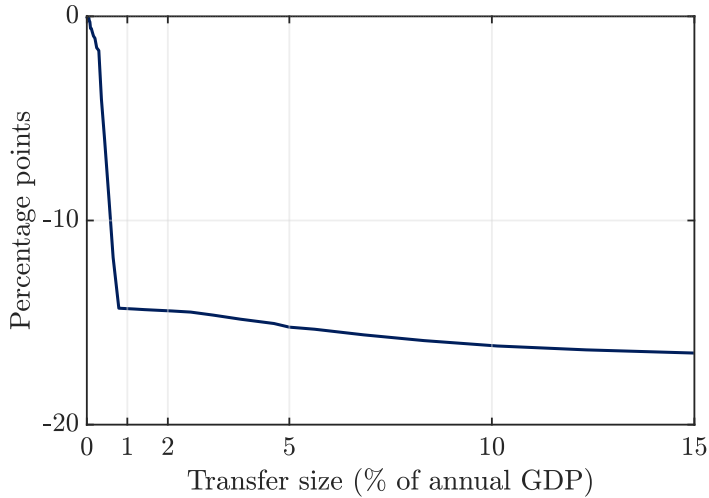
Both terms are negative $\Rightarrow$ $\mathcal{C}''(0) < 0$: the aggregate response is concave in the size of the transfer.

Extensive vs. intensive margin →

Change in share of constrained households



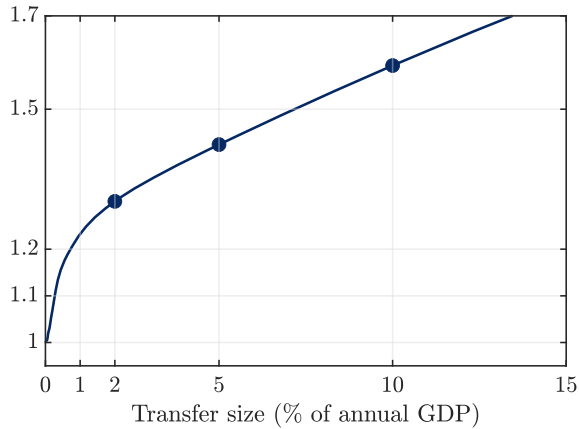
Taking stock

- Linear solution, changes in constrained share have zero first-order effects
  - Effectively, as if constrained households permanently MPC=1
- Non-linear solution:
  - Exensive margin: When a transfer moves households off the constraint, MPC drops discontinuously
  - Intensive margin: MPC is decreasing in wealth for unconstrained households

$\Rightarrow$ Aggregate MPC decreasing in transfers

Robustness of Partial Equilibrium Results

Linear/Non-linear Consumption Response



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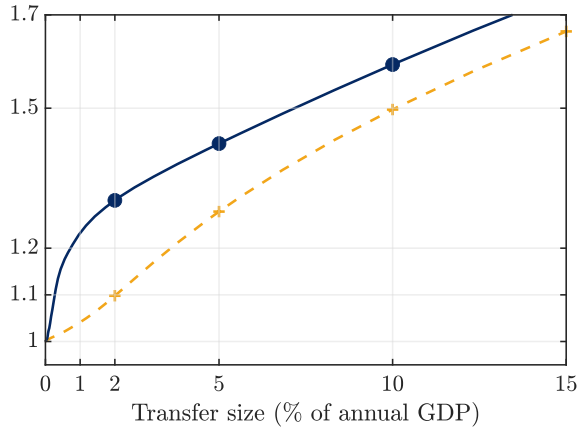

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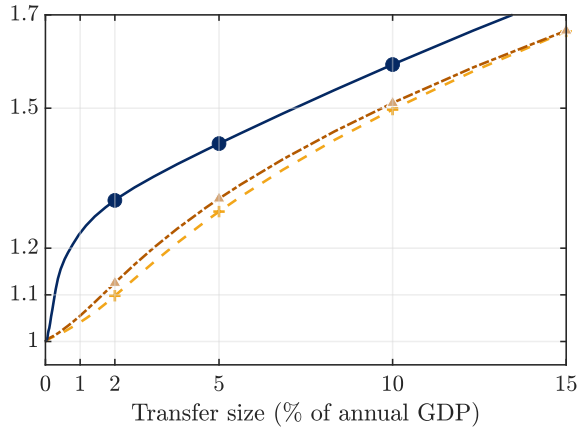

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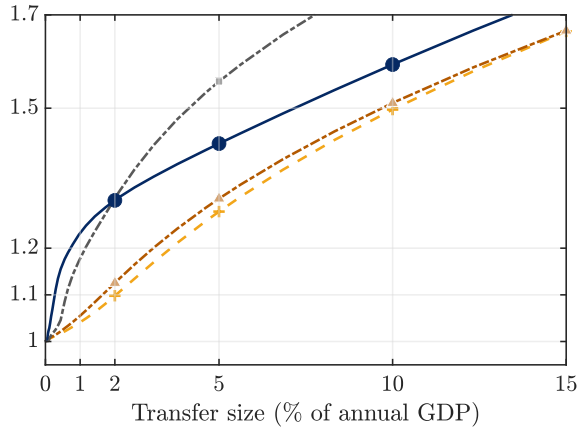

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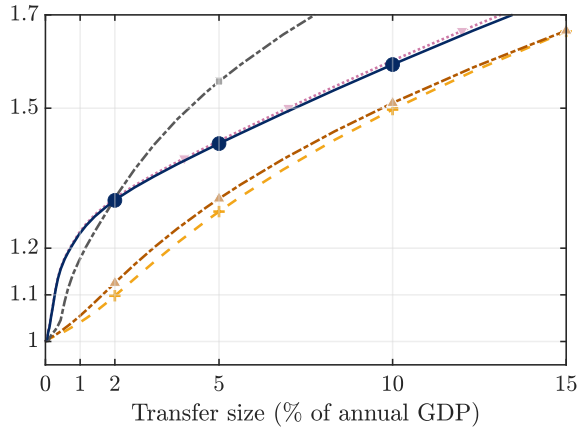

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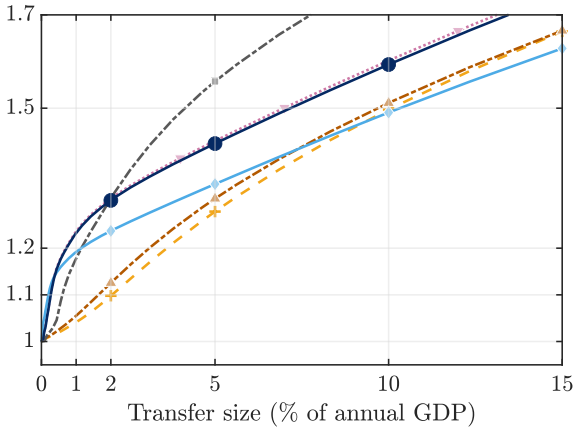

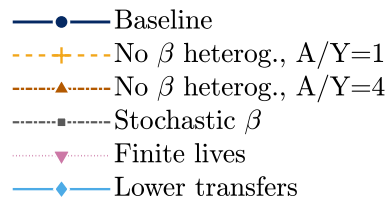

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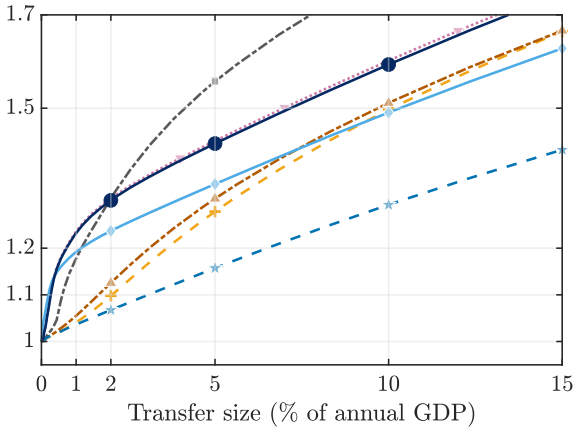

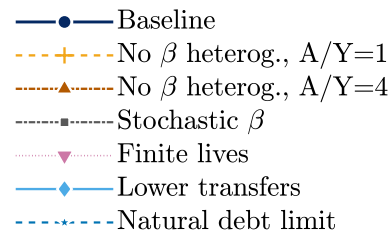

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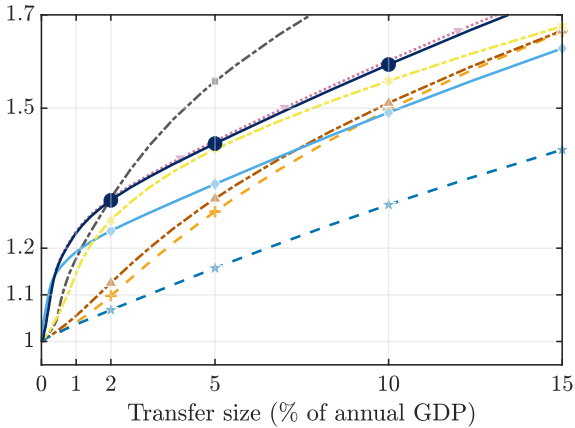

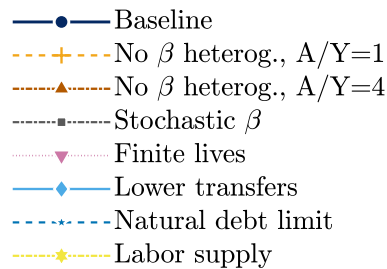

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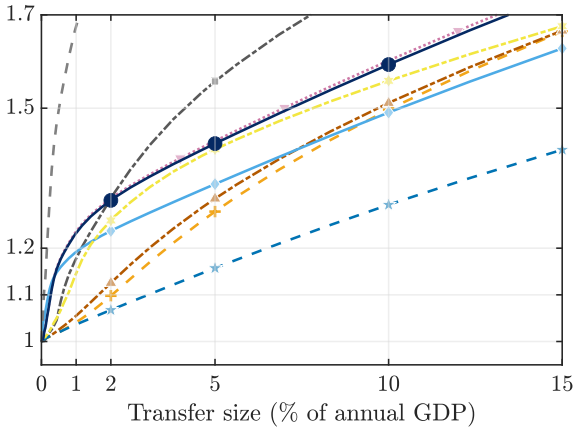

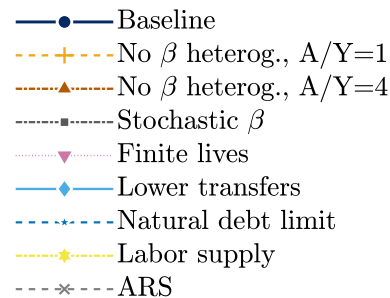

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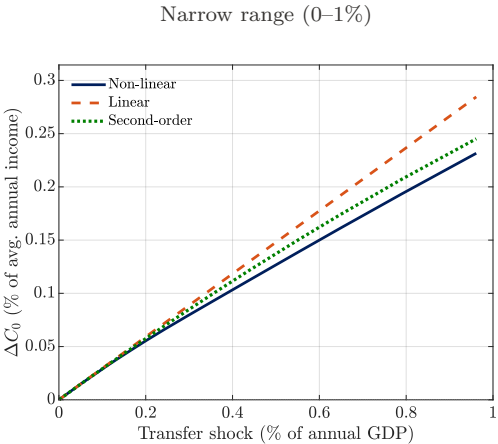

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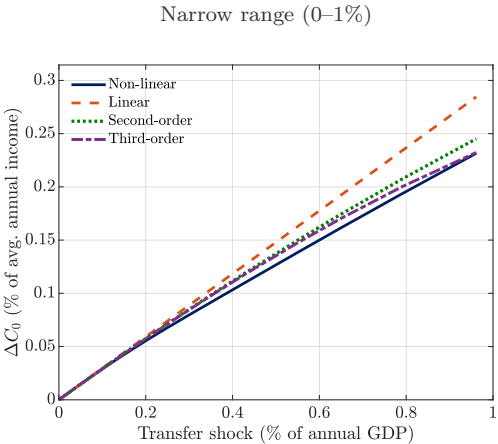
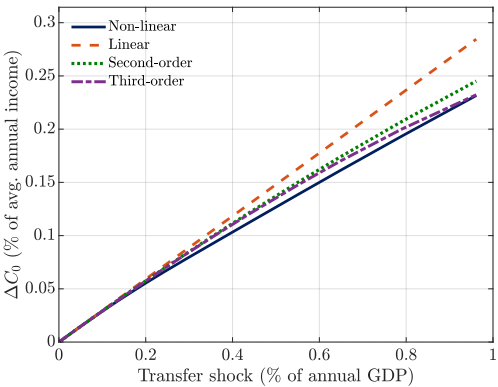
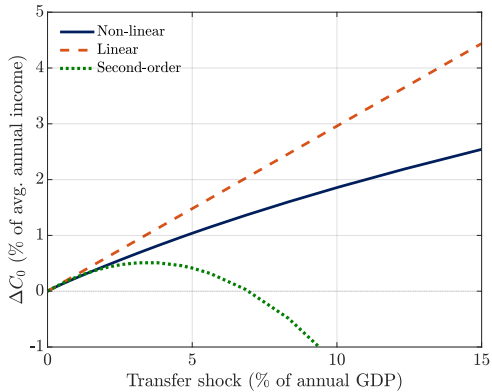


How about higher-order local methods?

Narrow range (0–1%)

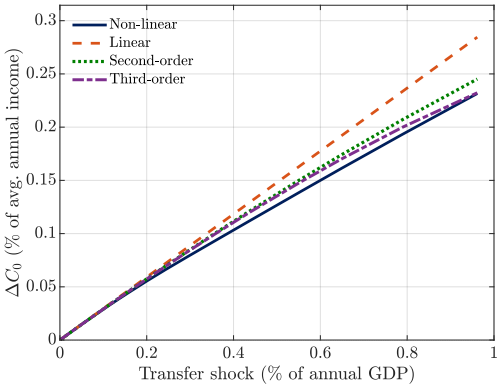


Full range (0–15%)

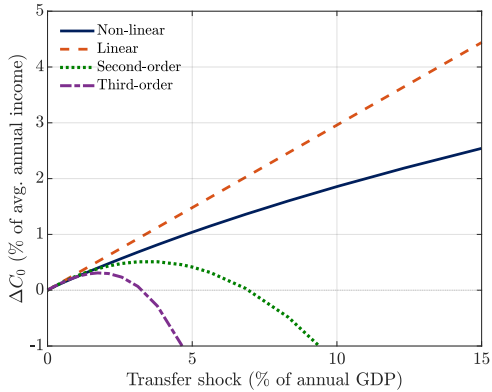


How about higher-order local methods?

Narrow range (0–1%)



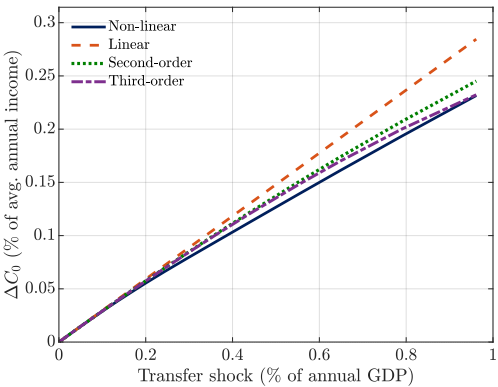
Full range (0–15%)



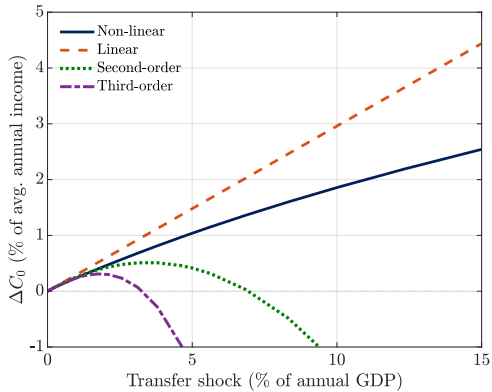
Second-order understates response for shocks $> 1\%$ because predicts that share of MPC=1 households fall at a linear rate

How about higher-order local methods?

Narrow range (0–1%)



Full range (0–15%)



Finite differences → Constrained vs. unconstrained → Decomposition →

Roadmap

1. Partial equilibrium model and calibration
2. Fiscal stimulus in general equilibrium
3. Other shocks

General Equilibrium Model

PE + production, labor supply, nominal rigidities, fiscal/monetary rules.

General Equilibrium Model

- **Households:**
  - Utility flow, $u(c_{it}) - v(n_{it})$ with $v' > 0, v'' > 0$
  - Income $(1-\tau)\left(\frac{W_t}{P_t} e_{it} n_{it} + d_{i,t}\right)$

General Equilibrium Model

- **Households:**
  - Utility flow, $u(c_{it}) - v(n_{it})$ with $v' > 0, v'' > 0$
  - Income $(1-\tau)\left(\frac{W_t}{P_t} e_{it} n_{it} + d_{i,t}\right)$
  - Labor union with sticky wages (linearized)

$$\pi_t = \kappa \left(v'(N_t) - \frac{1-\tau}{C_t} \right) + \bar{\beta}\, \pi_{t+1}, \qquad \text{(Phillips Curve)}$$

General Equilibrium Model

- **Households:**
  - Utility flow, $u(c_{it}) - v(n_{it})$ with $v' > 0, v'' > 0$
  - Income $(1-\tau)\left(\frac{W_t}{P_t} e_{it} n_{it} + d_{i,t}\right)$
  - Labor union with sticky wages (linearized)

$$\pi_t = \kappa \left(v'(N_t) - \frac{1-\tau}{C_t} \right) + \bar{\beta}\, \pi_{t+1}, \qquad \text{(Phillips Curve)}$$

- **Firms and Technology:**

$$\left.\begin{array}{l} \text{Production: } Y_t = N_t \\ \text{Flexible prices} \end{array}\right\} \Rightarrow\ W_t = P_t, \quad d_{i,t} = 0$$

General Equilibrium Model

Government Policies

General Equilibrium Model

Government Policies

- Government budget:

$$B_t + \tau Y_t = G + T_t + (1 + r_{t-1})B_{t-1}$$

- Fiscal rule ($t > 0$)

$$T_t = T - \phi_B \bar{r}(B_{t-1} - \bar{B})$$

- Monetary policy:

$$i_t = \bar{i} + \phi_\pi(\pi_t - \bar{\pi})$$

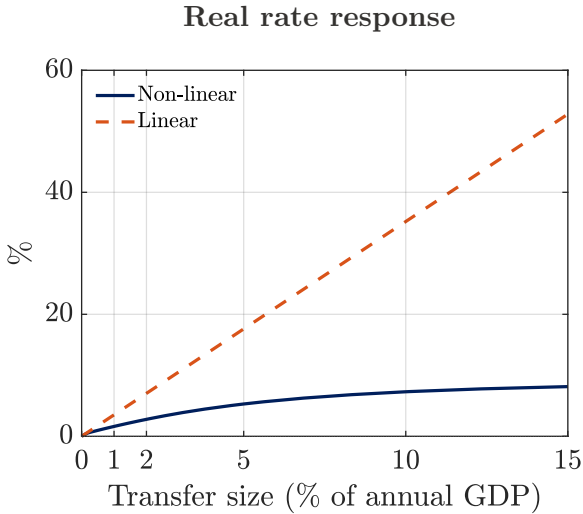
Calibration

Parameter	Value	Description
$\bar{B}$	4	Steady-state government debt=100% GDP
$\bar{r}$	0.02	Steady-state annual real interest rate
$\bar{\pi}$	0	Steady-state inflation
κ	0.01	Phillips curve slope
ν	1	Frisch elasticity of labor supply
ϕ_π	1.5	Taylor coefficient
ϕ_B	8	Fiscal rule
G	0.104	Government spending
T	0.15	Government transfers

Computation:

- Linear: sequence space Jacobian (ABRS)
- Non-linear: quasi-Newton algorithm

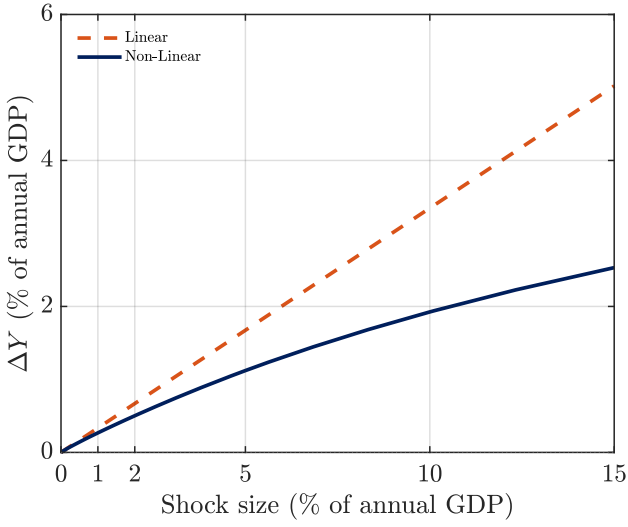
Flexible-Price Allocation: Real Interest Rate



- No output effects
- Linear methods overstate increase in demand and rise in r

With household labor choice →

HANK: Output Response

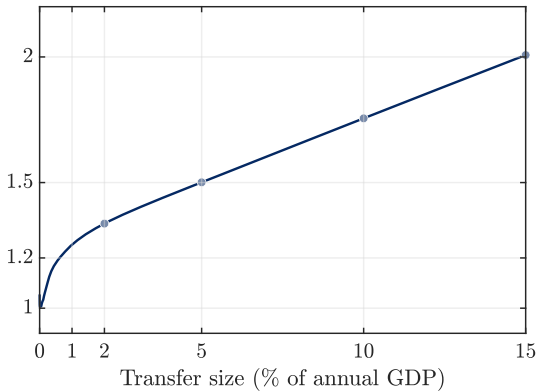


- At $\Delta = 2\%$ GDP: linear overstates by 34%
- At $\Delta = 5\%$ GDP: linear overstates by 50%

Impulse responses →

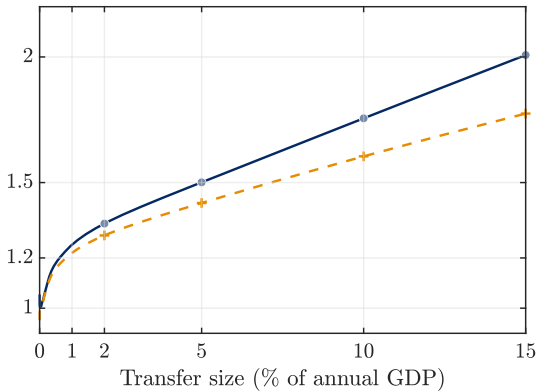
Robustness in HANK

Linear/Non-linear Output Response



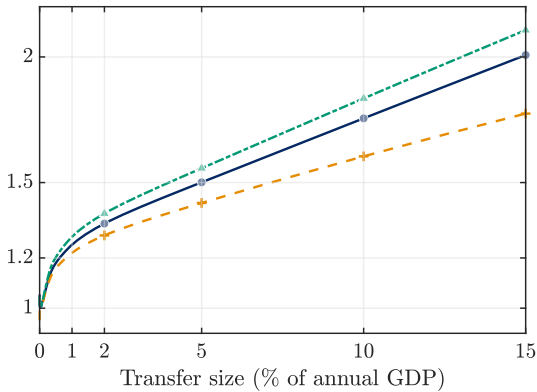
Robustness in HANK

Linear/Non-linear Output Response



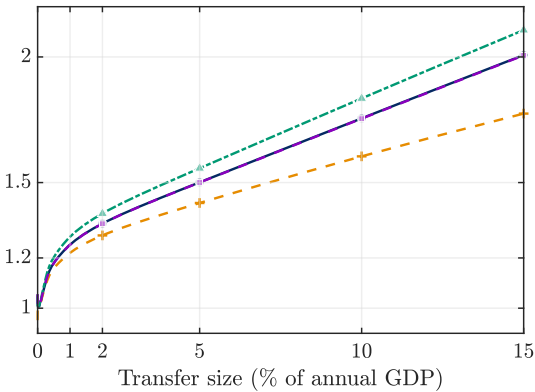
Robustness in HANK

Linear/Non-linear Output Response



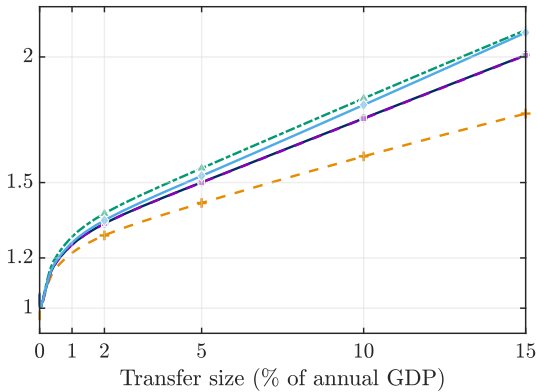
Robustness in HANK

Linear/Non-linear Output Response



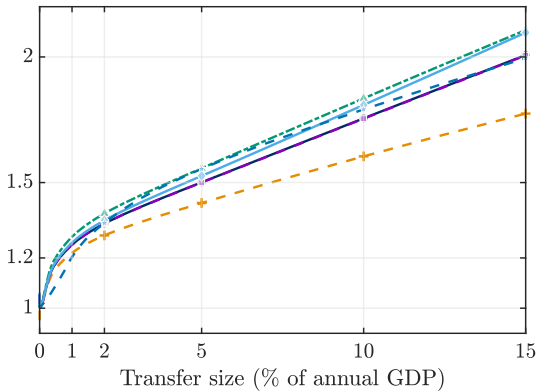
Robustness in HANK

Linear/Non-linear Output Response



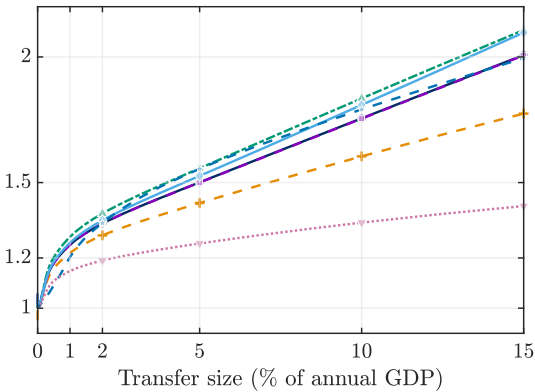
Robustness in HANK

Linear/Non-linear Output Response

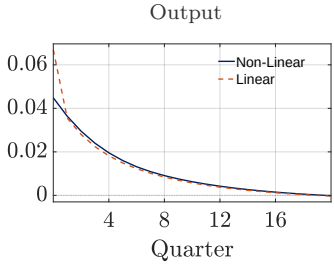


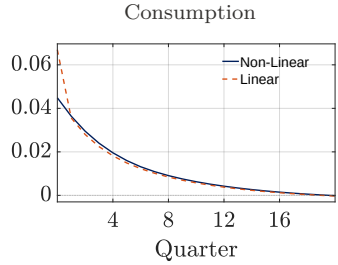
Robustness in HANK

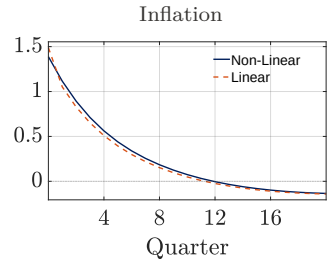
Linear/Non-linear Output Response

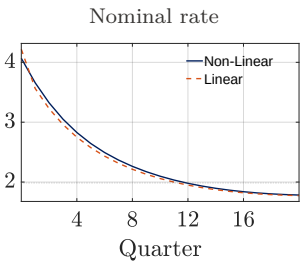


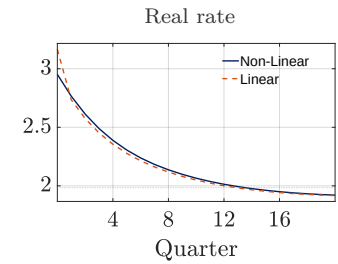
Linear method understates persistence

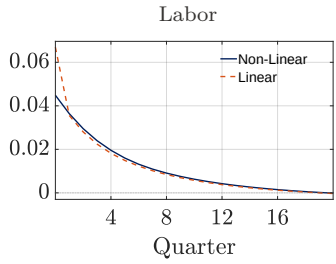










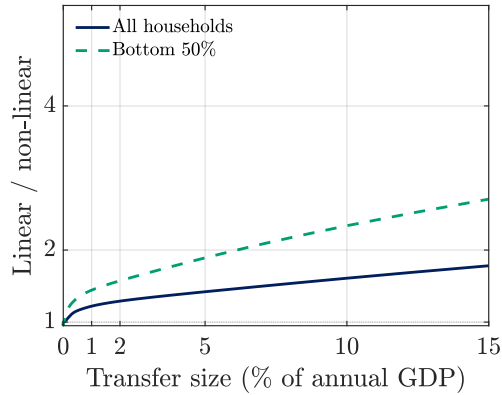


Note: Impulse Response to 5% Transfers

Targeted Transfers

Targeted Transfers

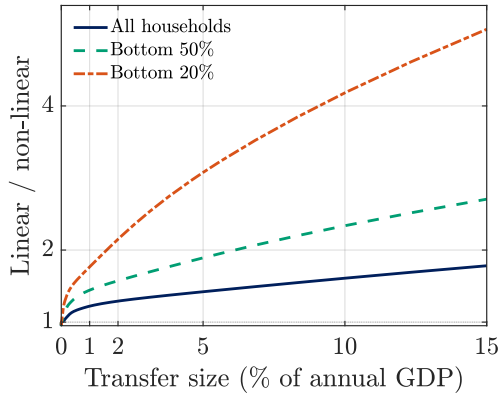
(b) Linear / non-linear



Note: GE output response to a one-time transfer directed to the bottom 20% (a) and the linear/non-linear ratio by recipient group (b).

Targeted Transfers

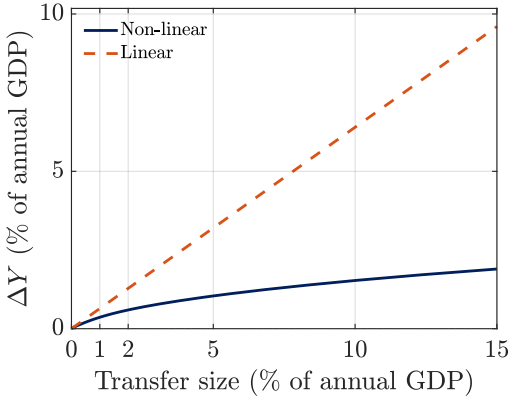
(b) Linear / non-linear



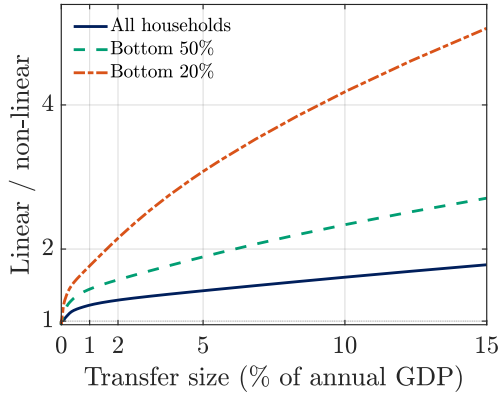
Note: GE output response to a one-time transfer directed to the bottom 20% (a) and the linear/non-linear ratio by recipient group (b).

Targeted Transfers

(a) Output response



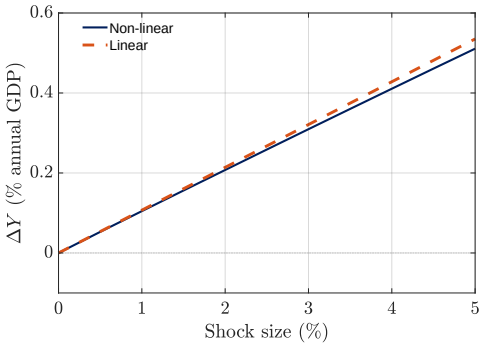
(b) Linear / non-linear



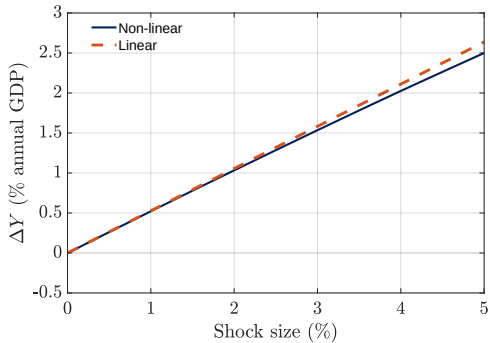
Note: GE output response to a one-time transfer directed to the bottom 20% (a) and the linear/non-linear ratio by recipient group (b).

Non-linearities for monetary policy and TFP

(a) Monetary (rate cut)



(b) TFP



▸ Tax-rate cut ▸ Discount-factor (β) shock

Conclusions

- HA models are valuable for analyzing fiscal and monetary policy because they capture realistic household behavior
- For monetary policy, linear approximation is accurate. But HA $\approx$ RA
- For fiscal stimulus: need non-linear methods
  - Works through departures from Ricardian equivalence, where MPCs are strongly non-linear in wealth:
  - First-order methods (SSJ) treat constrained households as permanently $MPC = 1$, overstating their true response. Higher-order local methods little improvement.

For local methods to be reliable for fiscal stimulus, "small" must be *very* small.

Appendix

Additional figures

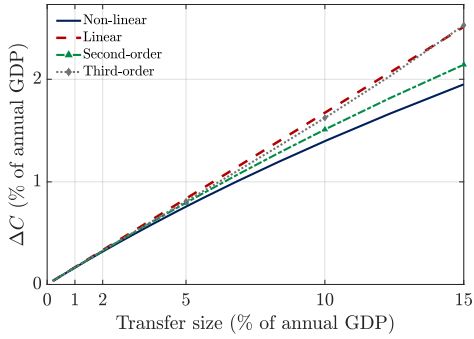
Implied parameters

Parameter	Value
Preferences & assets	
r (annual)	2%
$\bar{\beta}$ (mean disc. factor)	0.972
σ_β (dispersion)	0.010
Stochastic income process *(Kaplan–Violante 2022)*	
ϕ_z (AR persistence)	0.988
σ_η^2 (perm. var.)	0.044
σ_ε^2 (trans. var.)	0.638
λ (shock arrival)	1/4 per qtr

Back to calibration

Constrained vs. Unconstrained

Initially unconstrained

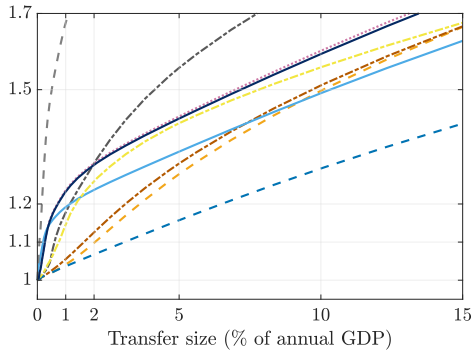


Note: Consumption response within each group, as a % of annual GDP per capita.

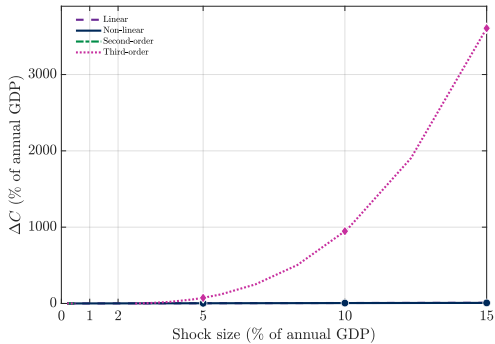
Back

PE: KMV calibration and first-year response

(a) KMV calibration

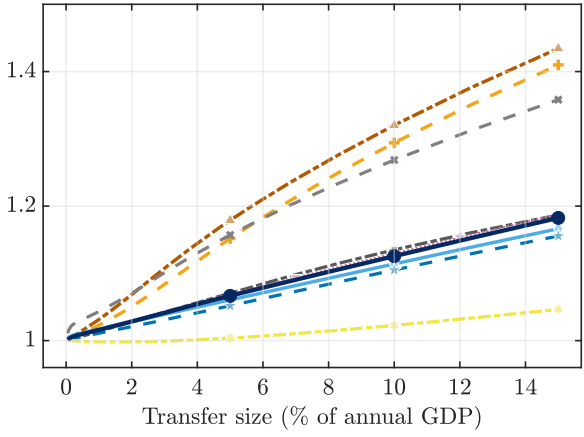


(b) First-year response



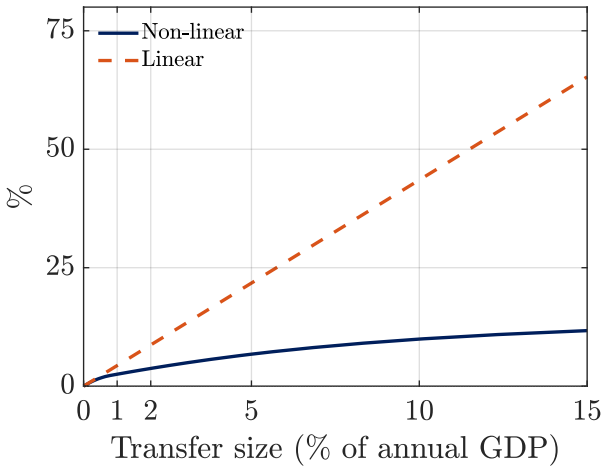
Note: Concavity is robust to the (Kaplan, Moll & Violante 2018) calibration and remains over the first year (panel b), though smaller than on impact.

GE: first-year linear-to-non-linear ratio



Note: Over the first year the linear method still overstates the output response, though by less than on impact.

GE: flexible prices with household labor choice

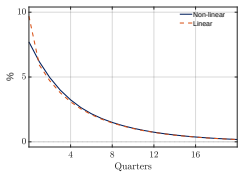


- Households choose labor freely (no union)
- Small output effects via labor-supply incentives
- Real-rate response remains concave in Δ — same mechanism as the union case

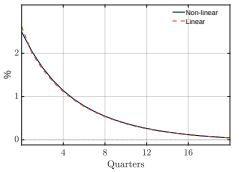
Back

GE: unfunded stimulus (FTPL) impulse responses

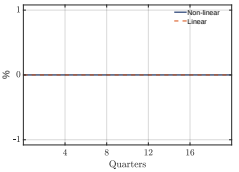
Output



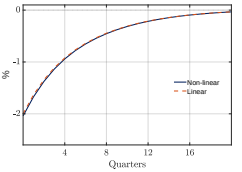
Inflation



Nominal rate

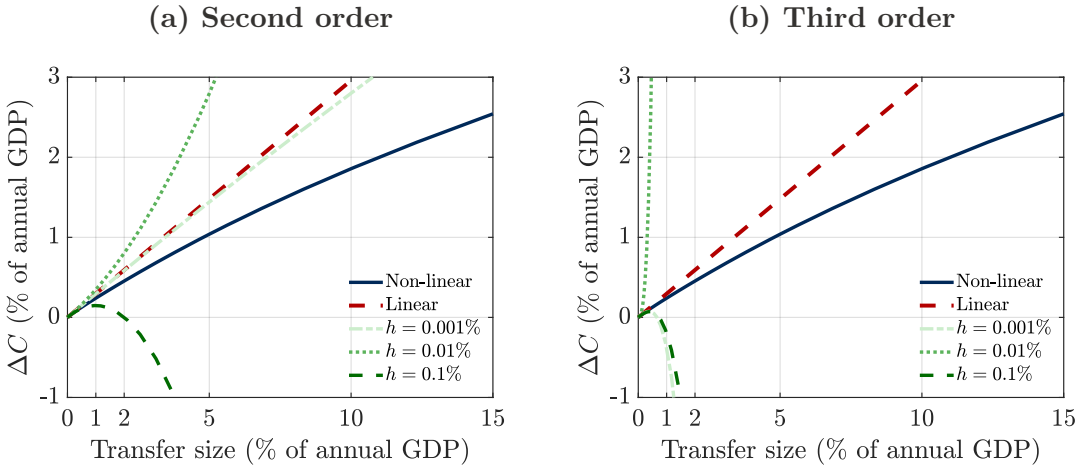


Real rate



Note: Under the FTPL (active fiscal, passive money), the output effect does not hinge on a failure of Ricardian equivalence, so non-linearity is much milder.

Higher-order with finite differences

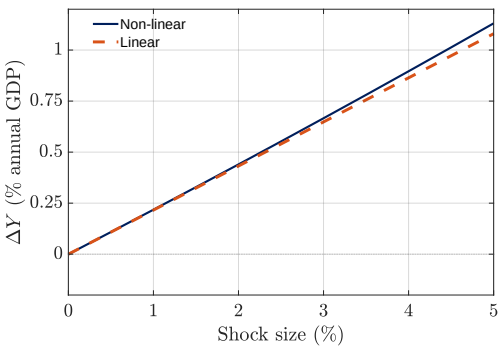


Extremely sensitive to step size and poorly behaved even for small shocks.

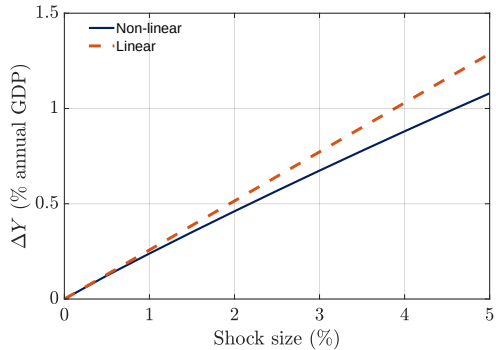
Back

Other shocks: discount factor and labor tax

(a) Discount-factor shock

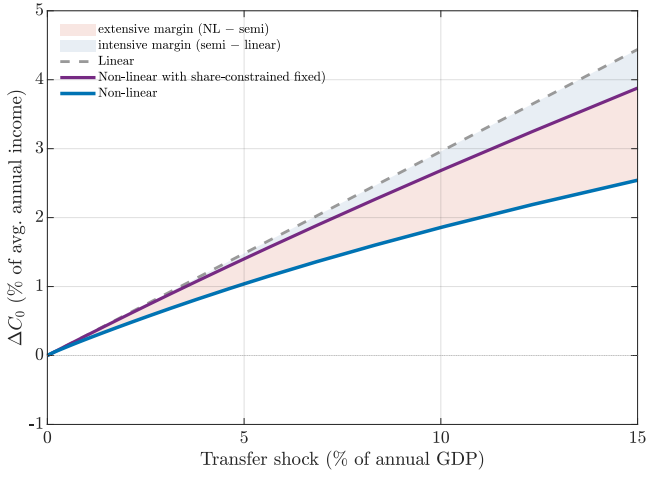


(b) Labor-tax cut



Note: Both generate milder non-linearity than lump-sum transfers; a τ cut shifts fewer resources to low-income, constrained households.

Extensive vs. intensive margin

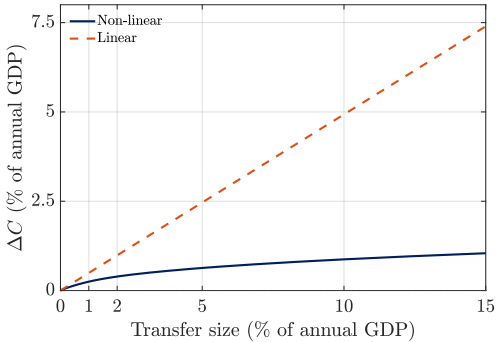


Note: impact PE

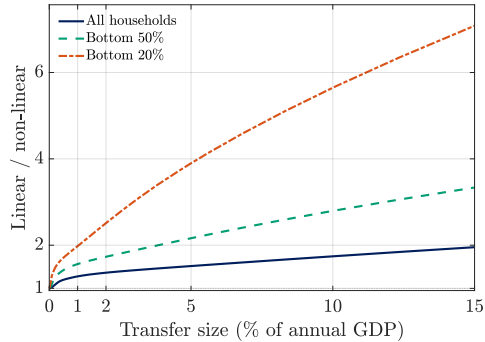
consumption response, decomposed into the intensive margin (curvature of unconstrained policies, holding the constrained share fixed) and the extensive margin (change in the

Targeted transfers: partial equilibrium

(a) Consumption response



(b) Linear / non-linear



Note: PE response to a one-time transfer directed to the bottom 20% (a) and the linear/non-linear ratio for All / Bottom 50% / Bottom 20% (b).