Conventional Inference in the Presence of Nonstationary Fractional Heavy Tailed Data

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Introduction

- IVX (cf. Phillips and Magdalinos, 2009; Kostakis, Magdalinos & Stamatogiannis, 2015) inference in parametric regressions of the form:

$$y_t = \mu + \beta f\left(x_{t-1}\right) + u_t, \quad f(.) \quad \textit{known}$$

- u_t is a m.d. error - finite second moment (main assumption)

- x_t is possibly nonstationary and heavy tailed (HT)
  - $x_t \sim I(d) \quad 1 - 1/\alpha < d < 2 - 1/\alpha$
  - Stability/tail parameter: $\begin{cases} 1 < \alpha \leq 2, & d \neq 1 \\ 0 < \alpha \leq 2, & d = 1 \end{cases}$

- If x_t is driven by finite variance innovations $\rightarrow \alpha = 2$
  - $x_t \sim I(d) \quad 1/2 < d < 3/2$

- Nonstationarity boundary: $d = 1 - 1/\alpha$ (nonstationary)

Introduction

- Datasets that exhibit HT behavior [e.g. Embrechts, Klppelberg & Mikosch (1997), Alder, Feldman & Taqqu (1998), Ibragimov, Ibragimov & Walden (2015)]:
  - actuarial science
  - economics
  - engineering
  - finance
  - meteorology
  - urban studies
  - epidemiology;
- Cohen, Davis, and Samorodnitsky (2022): COVID-19 deaths in US exhibit infinite variance.

Introduction: Examples in Economics

- **Exchange rates, exchange rate returns in emerging markets**: e.g. Falk and Wang (2003), Ibragimov, Ibragimov & Walden (2015), Nicolau & Rodrigues (2019).

- **Inflation**: Charemza, Hristova & Burridge (2006): monthly CPI/RPI Tail estimates ($\widehat{\alpha}$)

Country	Period	Price Innovations	Inflation Innovations
Mexico	1950--2000	1.27	1.28
Spain	1957--2000	1.43	1.40
UK	1956--2000	1.48	1.58
Canada	1950--2000	1.77	1.78
France	1957--2000	1.76	1.76
USA	1950--2000	1.90	1.91
Belgium	1957--2000	1.90	1.83

Introduction: Examples in Economics

- **Commodity Prices**: Barigozzi, Cavaliere & Trapani (2021): Copper, Gold Brent crude, Dubai Crude, Nickel, Cobalt

- **Realised Stock Return Volatility**: Kim & Meddahi (2020): Infinite variance, in particular during crisis periods.

Introduction

- Two main challenges for Inference in this framework

(1) FCLTs do not apply for nonstationary fractional processes when

$$1 - 1/\alpha \leq d < 1 \quad \text{and} \quad \alpha < 2$$

  - New asymptotic theory is required to characterize limit distributions

(2) Inference in non-stationary regressions is **non-conventional**

  - **nuisance parameters** feature in limit distributions e.g.
    - c: near-to-unity parameter
    - d: memory
    - α,σ,b: tail, scale, skewness
  - We demonstrate that IVX type of test statistics provide conventional inference -free of nuisance parameters
  - Practitioners can be agnostic about the **stationarity** and **tail** properties of the covariates

Introduction

- In general, inference is non-conventional when x_t is nonstationary
- Culprit: **endogeneity** i.e. the dependence of x_t (regressor) and u_t (regression error).
- e.g. Predictive/reduced form regression (weak endogeneity)

$$y_t = \mu + \beta x_{t-1} + u_t, \;\; E(u_t \mid x_{t-1}) = 0$$

- e.g. Structural regression (strong endogeneity)

$$y_t = \mu + \beta x_t + u_t$$

- Consider the test hypothesis $H_0 : \beta = 0$. Unless x_{t-1} is exogenous

$$\hat{t}_{OLS} \nrightarrow_d N(0,1)$$

- parametric estimators (e.g. OLS)

	weak endogeneity	strong endogeneity
stationarity	consistent estimators; conventional inference	inconsistent estimators; inference N/A
nonstationarity	consistent estimators; non-conventional inference	consistent estimators; non-conventional inference

Introduction: IV inference in parametric models

$$\hat{\beta}_{IV} = \frac{\sum_{t=1}^{n} Z_{t-1} y_t}{\sum_{t=1}^{n} f(x_{t-1}) Z_{t-1}} = \beta + \frac{\sum_{t=1}^{n} Z_{t-1} u_t}{\sum_{t=1}^{n} f(x_{t-1}) Z_{t-1}}$$

- Conventional inference applies if limit theory is (mixed) normal.
  - i.e. if a martingale Central Limit Theorem applies.
- Conventional IV inference through signal attenuation:
  - IVX (e.g. Phillips & Magdalinos, 2009; current work): quasi-differencing of x_t
  - CTLS (Hu, Kasparis & Wang, 2025): chronological trimming.
  - Smooth IV (Kasparis, Wang & Phillips, 2026): spatial trimming.

Preliminaries: HT distributions

- **DoA**(α): Domain of attraction of an α-stable law with stability parameter $\alpha \in (0, 2]$.

- A r.v. ξ is in the **DoA**(α) with $\alpha \in (0, 2)$ iff as $x \to \infty$

$$P(|\xi| > x) \sim cons.x^{-\alpha} h(x), \;\; h \text{ slowly varying}$$

- tail parameter $< 2 \Rightarrow$ tail parameter $=$ stability parameter
  - Tail condition holds with $\alpha = 2 \Rightarrow \xi \sim \text{DoA}(2)$.
  - $\xi \sim$ **DoA**$(2) \not\Rightarrow$ tail condition holds with $\alpha = 2$

- $\alpha < 2 \Rightarrow Var(\xi) = \infty$ [$\alpha < 1 \Rightarrow E(\xi)$ does not exist]

- $\alpha = 2 \Rightarrow Var(\xi) = \infty$ or $Var(\xi) < \infty$

HT fractional processes

- x_t: Heavy Tailed Nonstationary Fractional Process of memory d

$$1 - 1/\alpha < d < 2 - 1/\alpha.$$

- In this work

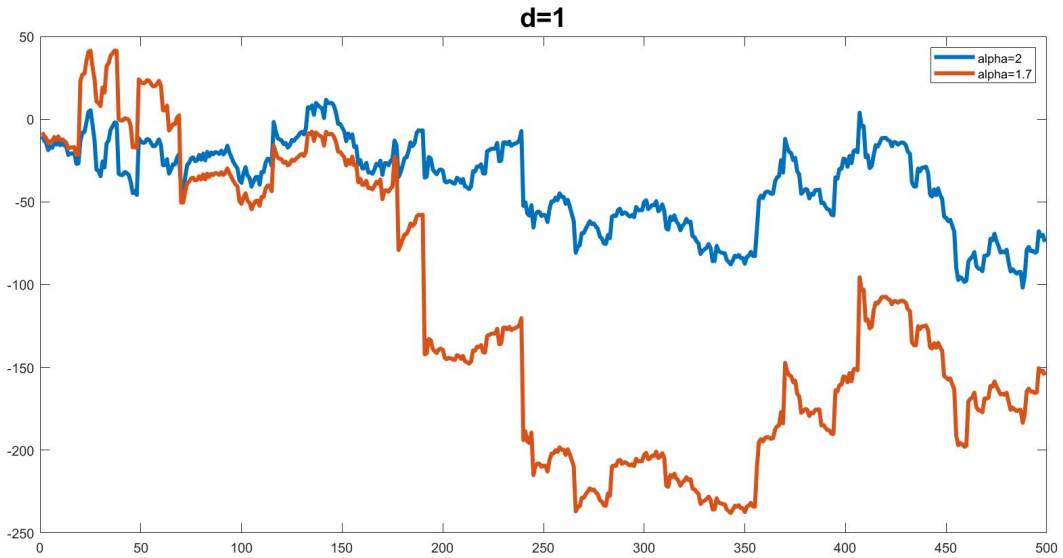
$$\alpha \in \begin{cases} (0, 2], \text{ if } d = 1 \\ (1, 2], \text{ if } d \neq 1 \end{cases}$$

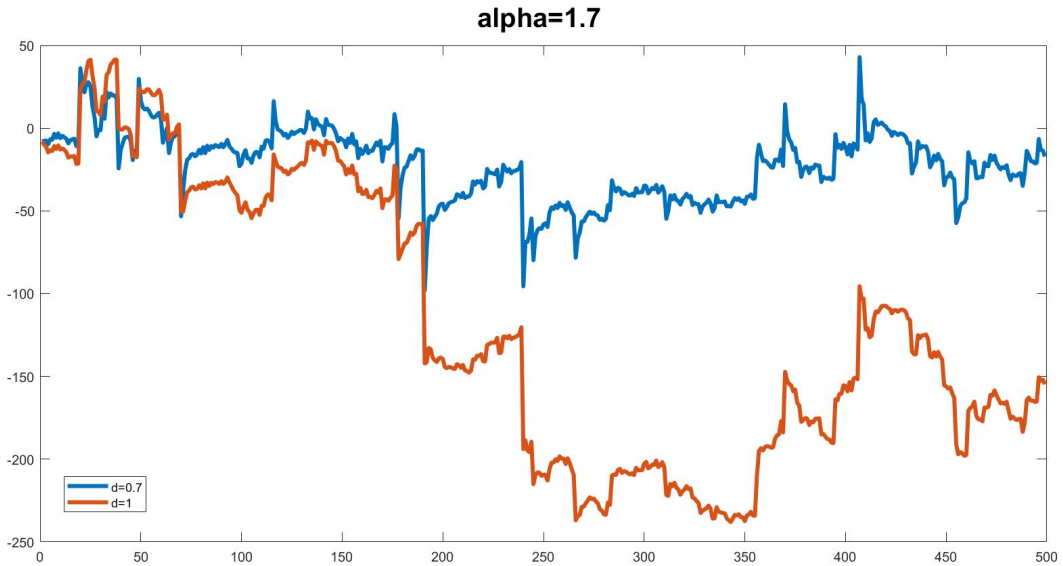
- $Var(x_t) < \infty \implies \alpha = 2 \implies 1/2 < d < 3/2.$

- HT - I(d) (type-I): $x_t = x_{t-1} + v_t$
  - $(1 - L)^\delta v_t = \xi_t$, $\xi_t \sim$ iid - DoA(α), $-1/\alpha < \delta < 1 - 1/\alpha$.
  - v_t: stationary HT - I(δ),
    - i.e. $v_t = \sum_{j=0}^{\infty} c_j \xi_{t-j}$,

- $d = 1 + \delta$

HT fractional processes





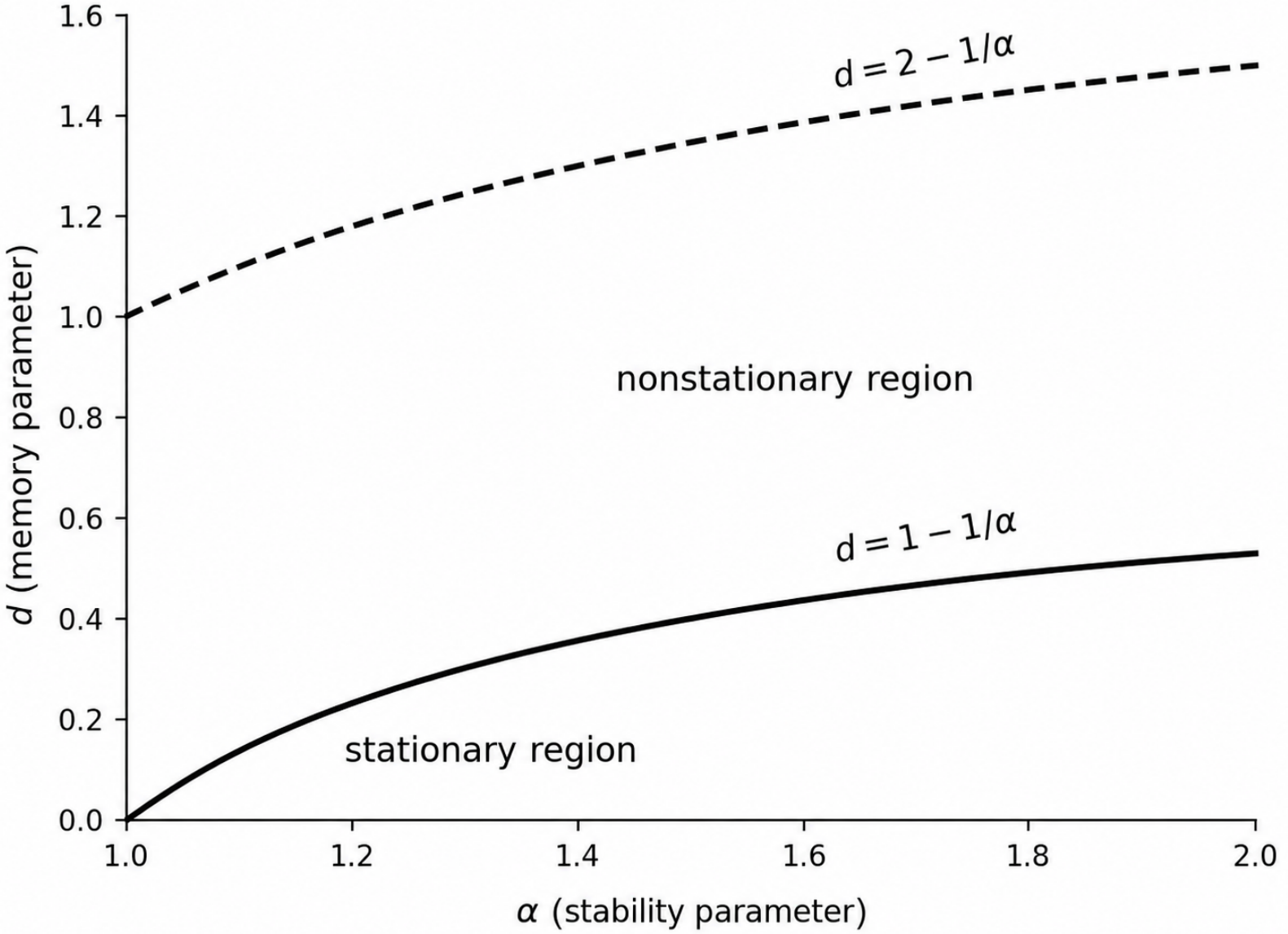
- No general limit theory for nonstationary HT data: e.g. FCLTs are not valid for $d < 1$, $\alpha < 2$ [limit processes are not càdlàg].

- **IVX** instruments: **FCLTs not satisfied**
  - IVX instruments are **Mildly Integrated** (MI) processes.
  - Amenable to the asymptotic machinery of Duffy and Kasparis (2021) for **Weakly Nonstationary Processes (WNPs)**

- Inference/limit distributions under nonstationarity are non standard.
  - This is due to endogeneity.
  - Use robust IV methods: IVX

- Heavy Tails enlarge the nonstationary region.
  - nonstationarity boundary: $d = 1 - 1/\alpha$

Stationary vs Nonstationary Region



A Classification of HT Processes

- **HT-SNPs** (strongly nonstationary processes):
  - $I(d)$, $1 - \frac{1}{\alpha} < d < 2 - \frac{1}{\alpha}$
  - NI, $x_t = (1 + \frac{c}{n})x_{t-1} + v_t$ (with v_t possibly fractional)
- **HT-WNPs** (weakly nonstationary processes):
  - **Mildly Integrated** (e.g. Giraitis & Phillips, 2006; Phillips & Magdalinos, 2007)

$$x_t = \left(1 - \frac{1}{\kappa_n}\right) x_{t-1} + v_t,\ \kappa_n^{-1},\ \kappa_n / n \to 0$$

  - **Boundary Fractional** $I(d)$, $d = 1 - \frac{1}{\alpha}$
- **HT-SPs** (stationary processes)
  - $I(d)$, $d < 1 - \frac{1}{\alpha}$

IVX Filtering

- In this project 5 types of HT-WNPs:
  - Boundary fractional Type I, $d = 1 - 1/\alpha$
  - Boundary fractional Type II, $d = 1 - 1/\alpha$
  - MI - short memory errors (MI-SM), IVX instrumentation
  - MI - long memory errors (MI-LM), IVX instrumentation
  - MI - negative memory errors (MI-NM), IVX instrumentation

IVX Filtering

- IVX instruments:

$$z_t = \sum_{j=0}^{t} \rho_n^{t-j} \Delta x_j,\ \rho_n = 1 - \frac{1}{\kappa_n},\ \kappa_n = o(n).$$

- $x_t \sim \text{SNP} \Rightarrow z_t \sim \text{MI}$

$$\left.\begin{array}{l} x_t \sim 1 - \frac{1}{\alpha} < d < 1 \\ x_t \sim d = 1 \\ x_t \sim 1 < d < 2 - \frac{1}{\alpha} \end{array}\right\} \text{IVX} \underset{\Rightarrow}{\text{filtering}} \left\{\begin{array}{l} z_t \sim \textit{MI-NM} \\ z_t \sim \textit{MI-SM} \\ z_t \sim \textit{MI-LM} \end{array}\right.$$

IVX Filtering

- IVX orders of magnitude/convergence rates:
  - s_n slowly varying sequence
  - $\mathcal{H} = d - (1 - 1/\alpha)$, $(\text{Hurst index}, 0 < \mathcal{H} < 1)$

$$\frac{x_n}{n^{\mathcal{H}} s_n} \xrightarrow{d} \alpha\text{-stable} \quad \& \quad \frac{z_n}{\kappa_n{}^{\mathcal{H}} s_{\kappa_n}} \xrightarrow{d} (\alpha\text{-stable})'$$

- Parametric convergence rate (linear regressions): $\sqrt{n} \cdot \left(n^{\mathcal{H}} s_n\right)$
  - $n^{\mathcal{H}} s_n$: regularly varying in n of order $\mathcal{H}$
- IVX convergence rate (linear regressions): $\sqrt{n} \cdot \left(\kappa_n{}^{\mathcal{H}} s_{\kappa_n}\right)$
  - $\kappa_n{}^{\mathcal{H}} s_{\kappa_n}$: regularly varying in κ_n of order $\mathcal{H}$

IVX Filtering

- The validity of IVX in the fractional case is not obvious,e.g. Andersen & Varneskov (2021; Remark 9):

"This setting [IVX] presents a non-nested alternative to ours where, however, the dependent variable is restricted from exhibiting long-range dependence. ***Whereas the IVX inference is not Gaussian uniformly in the persistence regimes, the corresponding Wald significance test is robust to the specific form of persistence****, which may also differ across regressors."*

Asymptotics for HT Processes

- Most results concern unit root models i.e. $d = 1$
- Unit Root Autoregressions
  - e.g. Chan & Tran (1989), Knight (1989), Phillips (1991), Caner (1997,1998)
- Non standard distributions that involve nuisance parameters (e.g. tail)
  - Cavaliere, Georgiev & Taylor (2018): Unit root tests / bootstrap methods
  - Barigozzi, Cavaliere & Trapani (2021): Test for # common trends in HT I(1) series

Existing Asymptotics for HT Processes

- Model $y_t = \mu + \beta f(x_{t-1}) + u_t$, f known.
- WLS estimator (parametric)

$$\tilde{\beta}_{WLS} = \left[\sum_{t=1}^{n} w(x_{t-1}) f(x_{t-1})\right]^{-1} \sum_{t=1}^{n} w(x_{t-1}) y_t$$

  - see also Samorodnitsky, Kurz-Kim & Rachev (2007).
- IVX (semi-parametric)

$$\hat{\beta}_{IVX} = \left[\sum_{t=1}^{n} w(z_{t-1}) f(x_{t-1})\right]^{-1} \left[\sum_{t=1}^{n} w(z_{t-1}) y_t\right]$$

  - z_t: IVX instrument

A Classification of HT Processes

- Weights: $|f(x)w(x)| = O\left(|x|^{\lambda}\right), \lambda \in (0, \alpha)$
  - e.g. $f(x) = x$
  - $w(.) = sign(.)$ [Cauchy Estimator; t-tests work for $0 < \alpha \leq 2$]
  - $w(.) = sign(.)\,|.|^{1/2}$ [t-tests work for $1 < \alpha \leq 2$]

New Asymptotics for HT Processes

- **Parametric Estimator**: Asymptotics for HT-SNPs i.e. $1 - 1/\alpha < d < 2 - 1/\alpha$

- Kasahara & Maejima (1988), Avram and Taqqu (1992)

$$\gamma_n^{*-1} x_{[nr]} \overset{fdd}{\to} \Lambda_{d,\alpha}(r), \;\; \gamma_n^* = n^{\mathcal{H}} s_n$$

  - Linear Fractional Stable Motion (LFSM)

$$\Lambda_{d,\alpha}(r) = \int_{-\infty}^{r} \left[(r-s)^{d-1} - (-s)^{d-1} 1_{\{s<0\}} \right] dZ_\alpha(s)$$

  - $Z_\alpha(s)$: Lévy Process
  - $\Lambda_{d,\alpha}(r)$ not cadlag for $1 - 1/\alpha < d < 1$, $0 < \alpha < 2$

New Asymptotics for HT Processes

Theorem 1. *Suppose that:*

(i) $\gamma_n^{*-1} x_{[nr]} \xrightarrow{fdd} \Lambda_{d,\alpha}(r)$;

(ii) $f : \mathbb{R} \to \mathbb{R}$ *satisfies the following. For some* $C_{\alpha'} \in (0, \infty)$,

$$|f(x)| \leq C_{\alpha'} \cdot \left(1 + |x|^{\alpha'}\right) \; 0 < \alpha' < \alpha,$$

$$|f(x) - f(y)| \leq C_{\alpha'} \cdot \begin{cases} |x|^{\alpha'-1}|y-x| + |y-x|^{\alpha'}, & 1 < \alpha' < \alpha, \; \alpha > 1, \\ |y-x|^{\alpha'}, & 0 < \alpha' < \alpha, \; \alpha \leq 1. \end{cases}$$

Then as $n \to \infty$,

$$\frac{1}{n} \sum_{t=1}^{n} f\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f\left(\Lambda_{\alpha,d}(r)\right) dr.$$

- A more general result in Kasparis, Wang & Phillips (in preparation)

New Asymptotics for HT Processes

Let

$$\tilde{u}_t = y_t - \tilde{\mu}_{WLS} - \tilde{\beta}_{WLS} f\left(x_{t-1}\right),$$

$$\tilde{\sigma}_u^2 = \frac{1}{n}\sum_{t=1}^{n} \tilde{u}_t^2$$

Corollary 2. $(f(x) = x)$ *As* $n \to \infty$,

$$\begin{bmatrix} \sqrt{n}\left(\tilde{\mu}_{WLS} - \mu\right) \\ \sqrt{n}\gamma_n^*\left(\tilde{\beta}_{WLS} - \beta\right) \end{bmatrix} = O_p(1)$$

and

$$\tilde{\sigma}_u^2 \to_p \sigma_u^2$$

New Asymptotics for HT Processes

- WNPs: FCLTs fail even if data have a second moment - see Duffy & Kasparis (2021).
- **HT-WNPs**: Let $0 < r, s \leq 1$.

$$x_t = x_t^+(\xi_t, ..., \xi_1) + x_t^-(\xi_0, \xi_{-1}, ...) =: x_t^+ + x_t^-$$

$$\gamma_n^{-1} x_{[nr]}^+ \to_{f.d.d.} X_r^+ \sim i.i.d.\, X^+ \sim \alpha\text{-stable}$$

$$\gamma_n^{-1}[x_{[nr]}^-, x_{[ns]}^-] \to_d [X^-, X^-],\ X^- \sim \alpha\text{-stable or zero}$$

Theorem 3. *Suppose that*

(i) $|f(x)| \sim |x|^{\alpha'}, \alpha' \in (0, \alpha)$;

(ii) x_t: $\mathcal{F}_t$*-measurable,* ***HT-WNP*** *[DoA(α) innovations];*

(iii) $\{u_t, \mathcal{F}_t\}$ **m.d.**, $\mathrm{E}\left(u_t^2 \mid \mathcal{F}_{t-1}\right) = \sigma_u^2 < \infty$;

(iv) *Additional technical conditions.*

$$\frac{1}{n}\sum_{t=1}^{n} f\left(\gamma_n^{-1} x_t\right) \xrightarrow{d} \int_{\mathbb{R}} f\left(x + X^{-}\right) \phi_{X^{+}}(x)\, dx,$$

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} f\left(\gamma_n^{-1} x_{t-1}\right) u_t \xrightarrow{d} MN\left(0, \sigma_u^2 \int_{\mathbb{R}} f\left(x + X^{-}\right)^2 \phi_{X^{+}}(x)\, dx\right),$$

$\phi_{X^{+}}$*: density of* X^{+} *and*

$$X^{+} \sim \alpha\text{-stable}, \;\; X^{-} \sim \alpha\text{-stable or zero}$$

New Asymptotics for HT Processes

- Scale (variance) parameter of X^+:
  - **MI-Long Memory** $(1 < d < 2 - 1/\alpha)$:

$$\int_0^\infty e^{-\alpha x} \left(\int_0^x e^y y^{d-2} dy \right)^\alpha dx =_{\alpha=2} \Gamma\left[2(3-d)\right] B\left[3-d, 2d-5\right].$$

  - **MI-Short Memory** $(d = 1$; see also Phillips & Magdalinos, 2009):

$$\frac{1}{\alpha} \left| \sum_{j=0}^{\infty} c_j \right|^\alpha .$$

  - **MI-Negative Memory** $(1 - 1/\alpha < d < 1)$:

$$\frac{1}{|d-1|^\alpha} \int_0^\infty \left| x^{d-1} - e^{-x} \int_0^x e^y y^{d-1} dy \right|^\alpha dx.$$

IVX t-statistic

- $H_0 : \beta = \beta_0$

$$\hat{t}_{IVX} = \sum_{t=1}^{n} w(z_{t-1}) f(x_t) \cdot \frac{\hat{\beta}_{IVX} - \beta_0}{\sqrt{\tilde{\sigma}_u^2 \sum_{t=1}^{n} w(z_{t-1})^2}}$$

  - $\tilde{\sigma}_u^2$ based on the parametric WLS residuals

Theorem 4. *Under* H_0, *as* $n \to \infty$

$$\hat{t}_{IVX} \xrightarrow{d} N(0, 1)$$

Simulations

- **Generate**:

$$\begin{bmatrix} v_t \\ \varepsilon_t \end{bmatrix} \sim \text{id. bivariate } t_\alpha \left[\Sigma_\delta\right]$$

  - Scale Matrix: $\Sigma_\delta = \begin{bmatrix} 1 & \delta \\ \delta & 1 \end{bmatrix}$, $|\delta| < 1$
  - $(I - L)^d x_t = v_t 1\{t > 0\}$
  - $u_t = sign(\varepsilon_t) |\varepsilon_t|^{\alpha/\nu}$, $\nu = 2.1$

- **Generate**: $y_t = \beta x_{t-1} + u_t$

- **Generate**: $\hat{t}_{IVX}$ for $H_0 : \beta = 0$

Simulations

Table 3: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.8, 2.1]$; **DGP1**; $[f(x),\ w(x)] = [x,\ x^{\langle 0.5 \rangle}]$; $d_* = 0.444$

		W-IVX					WLS					OLS				
n	d	δ=-0.99	-0.75	0.00	0.75	0.99	δ=-0.99	-0.75	0.00	0.75	0.99	δ=-0.99	-0.75	0.00	0.75	0.99
250	-0.156	0.074	0.077	0.081	0.077	0.081	0.075	0.078	0.081	0.079	0.082	0.055	0.060	0.064	0.059	0.055
500	-0.156	0.078	0.073	0.078	0.077	0.076	0.078	0.075	0.079	0.078	0.077	0.051	0.053	0.057	0.058	0.052
1000	-0.156	0.073	0.071	0.077	0.074	0.072	0.073	0.071	0.078	0.074	0.073	0.050	0.050	0.057	0.050	0.050
250	0.044	0.078	0.078	0.079	0.079	0.079	0.080	0.080	0.080	0.080	0.082	0.056	0.060	0.061	0.059	0.056
500	0.044	0.077	0.074	0.079	0.079	0.075	0.079	0.076	0.080	0.079	0.077	0.051	0.054	0.057	0.057	0.053
1000	0.044	0.074	0.073	0.078	0.072	0.073	0.076	0.073	0.079	0.072	0.074	0.051	0.050	0.058	0.048	0.049
250	0.244	0.077	0.078	0.079	0.079	0.078	0.089	0.086	0.083	0.088	0.090	0.059	0.062	0.063	0.062	0.059
500	0.244	0.070	0.072	0.076	0.077	0.073	0.080	0.079	0.079	0.083	0.082	0.053	0.057	0.060	0.059	0.057
1000	0.244	0.071	0.068	0.077	0.069	0.071	0.078	0.074	0.079	0.074	0.077	0.051	0.052	0.058	0.051	0.052
250	**0.444**	0.075	0.077	0.080	0.079	0.079	0.115	0.102	0.093	0.104	0.116	0.081	0.079	0.072	0.079	0.084
500	**0.444**	0.075	0.075	0.077	0.077	0.075	0.108	0.098	0.088	0.100	0.109	0.074	0.072	0.070	0.074	0.077
1000	**0.444**	0.071	0.071	0.073	0.069	0.071	0.102	0.093	0.080	0.087	0.099	0.068	0.066	0.065	0.063	0.067
250	0.644	0.078	0.080	0.084	0.079	0.078	0.176	0.141	0.107	0.140	0.174	0.155	0.126	0.093	0.125	0.156
500	0.644	0.072	0.074	0.081	0.077	0.074	0.164	0.136	0.102	0.135	0.168	0.143	0.121	0.091	0.120	0.147
1000	0.644	0.068	0.071	0.074	0.070	0.065	0.155	0.128	0.093	0.127	0.156	0.137	0.113	0.082	0.110	0.138
250	0.844	0.065	0.074	0.084	0.073	0.063	0.259	0.193	0.122	0.193	0.254	0.271	0.191	0.115	0.195	0.262
500	0.844	0.060	0.066	0.080	0.065	0.061	0.253	0.193	0.114	0.187	0.251	0.260	0.193	0.109	0.191	0.261
1000	0.844	0.052	0.063	0.075	0.063	0.056	0.245	0.183	0.106	0.182	0.246	0.252	0.183	0.101	0.182	0.252
250	1.044	0.058	0.068	0.087	0.068	0.057	0.327	0.233	0.133	0.235	0.324	0.358	0.243	0.126	0.246	0.348
500	1.044	0.056	0.062	0.081	0.059	0.055	0.325	0.231	0.123	0.229	0.326	0.352	0.240	0.116	0.241	0.350
1000	1.044	0.051	0.056	0.074	0.057	0.051	0.318	0.225	0.117	0.225	0.314	0.345	0.234	0.112	0.235	0.343

Simulations

Table 5: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.4, 2.1]$; **DGP1**; $[f(x),\ w(x)] = [x,\ x^{\langle 0.5 \rangle}]$; $d_* = 0.286$

		W-IVX					WLS					OLS				
n	d	δ=-0.99	-0.75	0.00	0.75	0.99	δ=-0.99	-0.75	0.00	0.75	0.99	δ=-0.99	-0.75	0.00	0.75	0.99
250	-0.314	0.065	0.069	0.075	0.071	0.068	0.067	0.070	0.076	0.073	0.070	0.045	0.050	0.051	0.049	0.043
500	-0.314	0.066	0.069	0.072	0.069	0.063	0.067	0.070	0.073	0.071	0.065	0.048	0.050	0.051	0.049	0.048
1000	-0.314	0.061	0.064	0.068	0.068	0.065	0.062	0.065	0.069	0.069	0.066	0.043	0.042	0.044	0.045	0.044
250	-0.114	0.065	0.069	0.074	0.071	0.065	0.069	0.072	0.076	0.073	0.069	0.046	0.050	0.050	0.050	0.044
500	-0.114	0.065	0.068	0.071	0.069	0.065	0.067	0.071	0.073	0.072	0.066	0.047	0.049	0.050	0.049	0.047
1000	-0.114	0.062	0.063	0.067	0.068	0.065	0.064	0.065	0.069	0.069	0.067	0.042	0.042	0.043	0.044	0.044
250	0.086	0.066	0.068	0.073	0.071	0.064	0.073	0.077	0.076	0.078	0.073	0.048	0.050	0.051	0.052	0.044
500	0.086	0.066	0.066	0.069	0.069	0.064	0.073	0.071	0.073	0.076	0.070	0.047	0.048	0.050	0.049	0.048
1000	0.086	0.062	0.064	0.065	0.065	0.063	0.067	0.068	0.069	0.068	0.068	0.041	0.043	0.043	0.044	0.043
250	**0.286**	0.067	0.074	0.072	0.070	0.067	0.092	0.092	0.085	0.090	0.093	0.052	0.056	0.055	0.056	0.050
500	**0.286**	0.067	0.069	0.067	0.072	0.067	0.088	0.084	0.079	0.088	0.087	0.051	0.050	0.053	0.053	0.048
1000	**0.286**	0.063	0.066	0.068	0.062	0.059	0.085	0.078	0.076	0.078	0.077	0.045	0.046	0.048	0.045	0.045
250	0.486	0.073	0.077	0.074	0.074	0.072	0.132	0.119	0.096	0.113	0.132	0.078	0.080	0.069	0.076	0.078
500	0.486	0.068	0.069	0.070	0.076	0.068	0.123	0.107	0.091	0.110	0.124	0.075	0.069	0.066	0.073	0.072
1000	0.486	0.067	0.067	0.070	0.066	0.061	0.119	0.103	0.090	0.101	0.108	0.070	0.066	0.062	0.064	0.061
250	0.686	0.067	0.077	0.077	0.072	0.069	0.191	0.160	0.116	0.150	0.193	0.157	0.133	0.098	0.127	0.158
500	0.686	0.064	0.067	0.071	0.073	0.064	0.181	0.146	0.104	0.150	0.185	0.147	0.121	0.088	0.127	0.149
1000	0.686	0.061	0.064	0.070	0.065	0.057	0.175	0.141	0.103	0.145	0.166	0.146	0.116	0.087	0.118	0.132
250	0.886	0.054	0.067	0.077	0.065	0.057	0.261	0.203	0.133	0.197	0.263	0.256	0.199	0.123	0.189	0.259
500	0.886	0.053	0.059	0.069	0.059	0.051	0.254	0.190	0.121	0.193	0.248	0.247	0.183	0.113	0.188	0.245
1000	0.886	0.048	0.052	0.066	0.056	0.046	0.243	0.183	0.117	0.187	0.236	0.242	0.177	0.110	0.183	0.233

Simulations

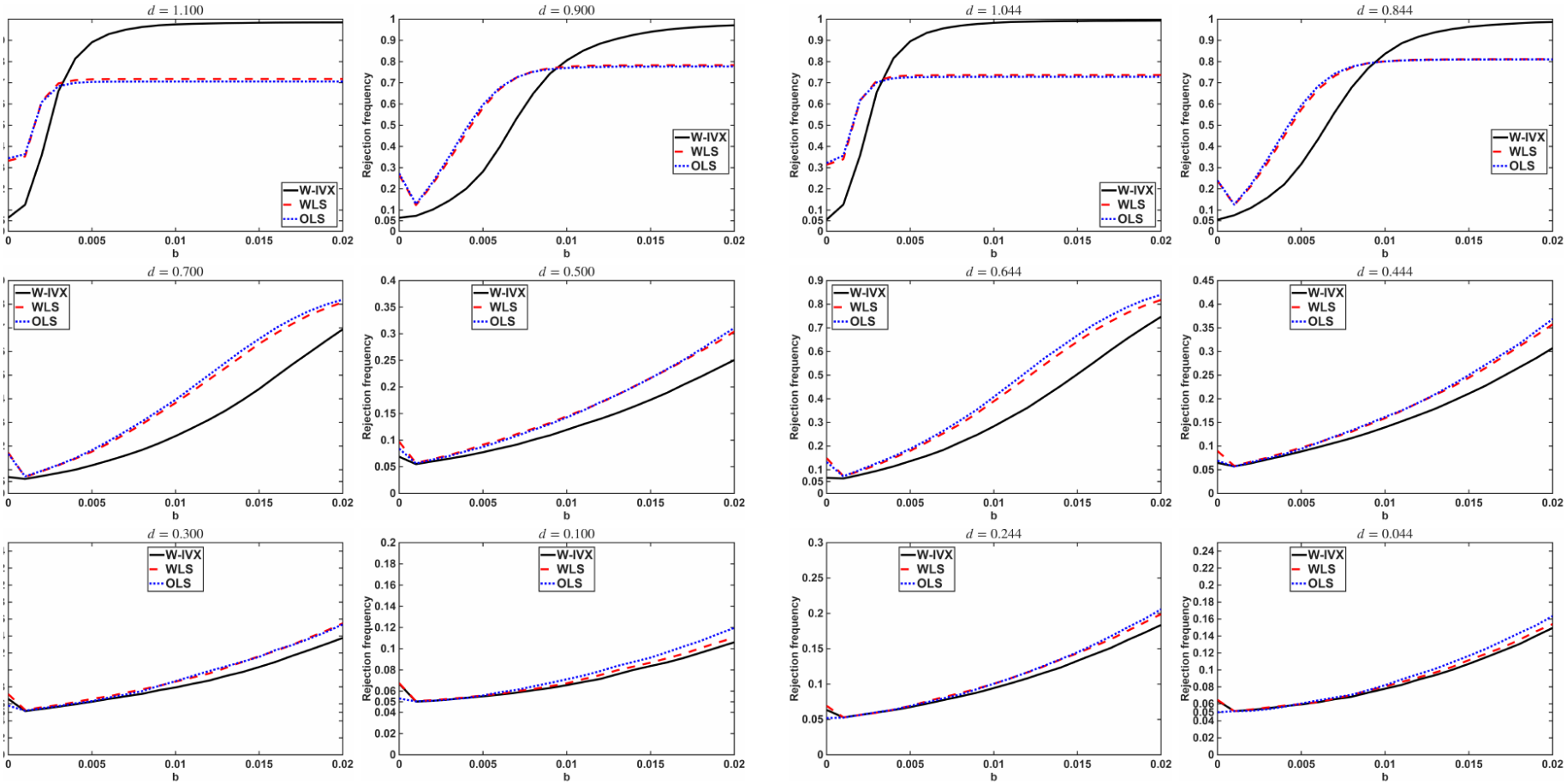
Table 11: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.8, 2.1]$; $f(x) = x$; $d_* = 0.444$; **DGP1**

$w(x)$		$x^{\langle 0 \rangle}$			$x^{\langle 0.5 \rangle}$			$x^{\langle 0.75 \rangle}$			$x^{\langle 1 \rangle}$		
n	d	δ=-0.99	0.00	0.99	δ=-0.99	0.00	0.99	δ=-0.99	0.00	0.99	δ=-0.99	0.00	0.99
250	-0.156	0.072	0.072	0.071	0.074	0.081	0.081	0.055	0.063	0.055	0.054	0.065	0.054
500	-0.156	0.067	0.066	0.069	0.078	0.078	0.076	0.051	0.058	0.049	0.052	0.057	0.054
1000	-0.156	0.065	0.066	0.065	0.073	0.077	0.072	0.050	0.057	0.048	0.049	0.055	0.047
250	0.044	0.073	0.068	0.072	0.078	0.079	0.079	0.053	0.063	0.055	0.055	0.064	0.054
500	0.044	0.067	0.069	0.070	0.077	0.079	0.075	0.053	0.057	0.047	0.052	0.058	0.053
1000	0.044	0.066	0.066	0.064	0.074	0.078	0.073	0.050	0.057	0.048	0.049	0.053	0.048
250	0.244	0.072	0.072	0.075	0.077	0.079	0.078	0.054	0.064	0.053	0.055	0.065	0.053
500	0.244	0.072	0.071	0.072	0.070	0.076	0.073	0.053	0.059	0.049	0.053	0.059	0.054
1000	0.244	0.068	0.067	0.067	0.071	0.077	0.071	0.050	0.058	0.049	0.050	0.055	0.047
250	**0.444**	0.084	0.077	0.082	0.075	0.080	0.079	0.059	0.068	0.059	0.058	0.069	0.058
500	**0.444**	0.079	0.074	0.078	0.075	0.077	0.075	0.057	0.064	0.054	0.057	0.064	0.060
1000	**0.444**	0.077	0.072	0.076	0.071	0.073	0.071	0.053	0.063	0.052	0.053	0.060	0.052
250	0.644	0.089	0.083	0.088	0.078	0.084	0.078	0.071	0.074	0.065	0.068	0.077	0.068
500	0.644	0.081	0.082	0.080	0.072	0.081	0.074	0.067	0.067	0.064	0.065	0.071	0.070
1000	0.644	0.077	0.075	0.073	0.068	0.074	0.065	0.064	0.068	0.063	0.064	0.068	0.066
250	0.844	0.082	0.086	0.079	0.065	0.084	0.063	0.071	0.075	0.066	0.068	0.079	0.071
500	0.844	0.072	0.082	0.070	0.060	0.080	0.061	0.063	0.071	0.064	0.063	0.073	0.065
1000	0.844	0.066	0.079	0.063	0.052	0.075	0.056	0.064	0.070	0.061	0.059	0.070	0.065
250	1.044	0.074	0.090	0.076	0.058	0.087	0.057	0.074	0.079	0.070	0.071	0.080	0.072
500	1.044	0.066	0.087	0.063	0.056	0.081	0.055	0.066	0.074	0.065	0.064	0.076	0.064
1000	1.044	0.059	0.082	0.057	0.051	0.074	0.051	0.063	0.071	0.063	0.059	0.075	0.067

Simulations

Figure 7: Size Adjusted Power (nominal size 5%): **DGP1**, $[f(x), w(x)] = [x, x^{\langle 0.75 \rangle}]$, $\delta = -0.95$, $n = 1000$

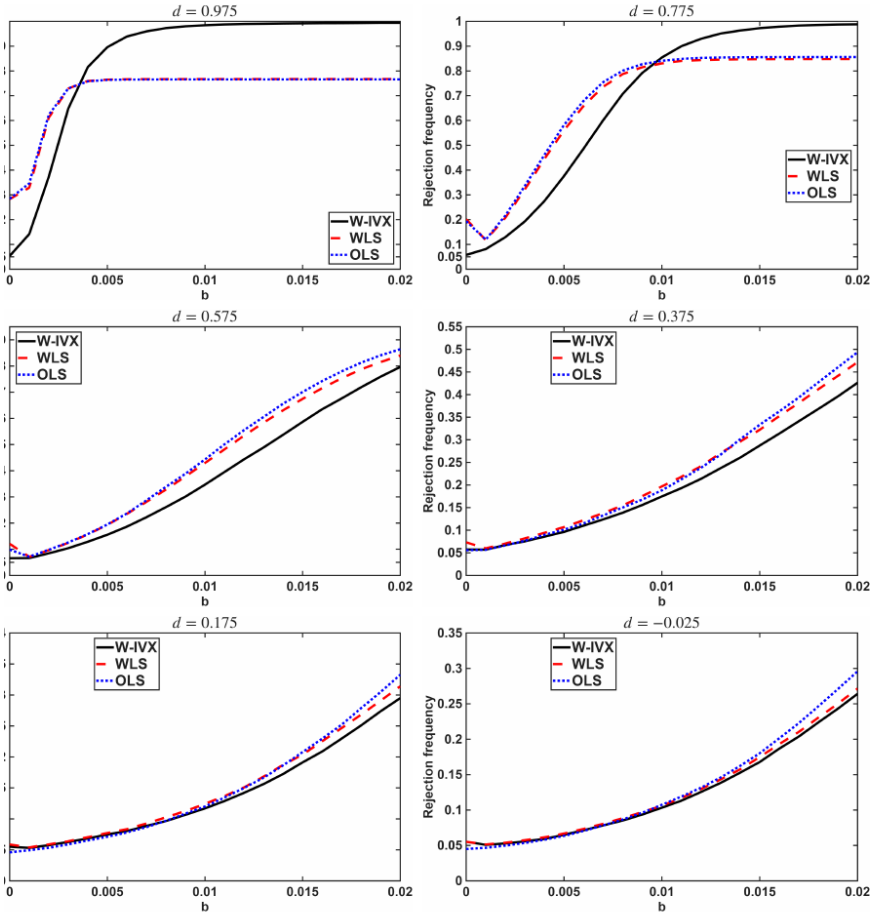
(a) $\alpha = 2$, $d_* = 0.5$ (b) $\alpha = 1.8$, $d_* = 0.444$



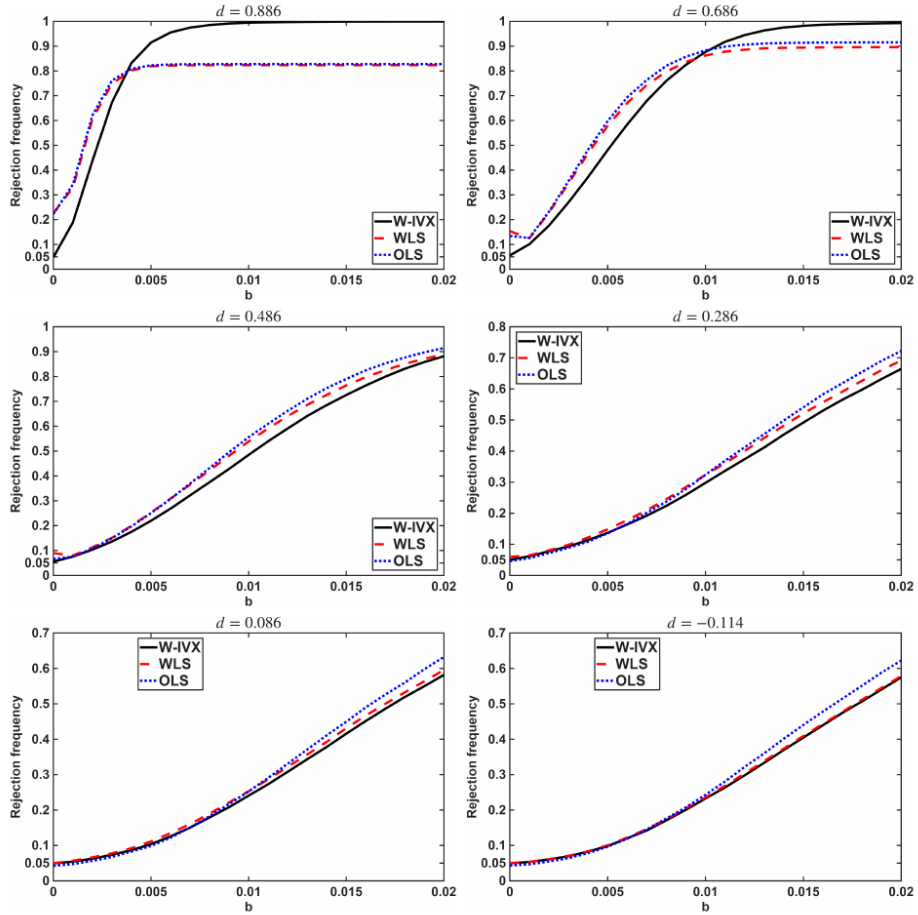
Simulations

Figure 7: Size Adjusted Power (nominal size 5%): $[f(x), w(x)] = [x, x^{\langle 0.75 \rangle}]$, $\delta = -0.95$, $n = 1000$ (continued)

(c) $\alpha = 1.6$, $d_* = 0.375$



(d) $\alpha = 1.4$, $d_* = 0.286$



Application: Predictability of Stock Returns

Table 33: Memory and Tail Estimates (1926–2024)

Panel A: Monthly Returns; 1926:01 – 2024:12

Dependent Variables	LW	ELW	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$
Returns	0.051	0.054	2.697	2.690	2.614	2.365
Returns-Rfree	0.055	0.058	2.774	2.670	2.602	2.348
CRSP_SPvw	-0.012	-0.011	2.658	2.885	2.772	2.567
CRSP_SPvw-Rfree	0.011	0.010	2.696	2.851	2.768	2.548

Panel B: Monthly Predictors; 1926:01 – 2024:12

Predictors	LW	ELW	$\hat{\alpha}_1$	d_{1*}	$\hat{\alpha}_2$	d_{2*}	$\hat{\alpha}_3$	d_{3*}	$\hat{\alpha}_4$	d_{4*}
BM	**0.943**	**0.951**	**1.707**	**0.414**	**1.879**	**0.468**	**1.916**	**0.478**	**1.812**	**0.448**
tbl	**1.080**	**1.085**	**1.899**	**0.473**	**1.762**	**0.432**	**1.527**	**0.345**	**1.439**	**0.305**
lty	**1.042**	**1.048**	2.628	0.500	2.391	0.500	2.228	0.500	**1.972**	**0.493**
ntis	**0.937**	**0.859**	2.011	0.500	**1.923**	**0.480**	**1.901**	**0.474**	**1.897**	**0.473**
Inflation	**0.516**	**0.538**	2.666	0.500	2.888	0.500	2.761	0.500	2.503	0.500
svar	**0.518**	**0.518**	**1.469**	**0.319**	**1.299**	**0.230**	**1.169**	**0.144**	**1.144**	**0.126**
DY	**0.886**	**0.906**	2.327	0.500	2.373	0.500	2.414	0.500	2.263	0.500
EP	**0.698**	**0.699**	2.308	0.500	2.103	0.500	2.129	0.500	2.065	0.500
Dpay	**0.657**	**0.630**	**1.754**	**0.430**	**1.943**	**0.485**	**1.968**	**0.492**	**1.726**	**0.421**

Panel C: Quarterly Returns; 1926:1 – 2024:4

Dependent Variables	LW	ELW	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$
Returns	-0.154	-0.135	2.639	2.696	2.464	2.291
Returns-Rfree	-0.140	-0.132	2.378	2.551	2.283	2.300
CRSP_SPvw	-0.309	-0.312	2.743	2.940	2.905	2.625
CRSP_SPvw-Rfree	-0.199	-0.204	3.020	2.937	2.680	2.554

Panel D: Quarterly Predictors; 1926:1 – 2024:4

Predictors	LW	ELW	$\hat{\alpha}_1$	d_{1*}	$\hat{\alpha}_2$	d_{2*}	$\hat{\alpha}_3$	d_{3*}	$\hat{\alpha}_4$	d_{4*}
BM	**0.739**	**0.741**	**1.885**	**0.470**	**1.681**	**0.405**	**1.787**	**0.440**	**1.808**	**0.447**
svar	**0.388**	**0.394**	**1.369**	**0.269**	**1.323**	**0.244**	**1.267**	**0.211**	**1.204**	**0.169**

Rebalancing Memory Tails in Predictive Regression

- A linear relationship between returns and risk measures does not seem implausible
  - misbalanced both in terms of memory and tails
- Previous work:
  - **balance memory using fractionally differenced predictors** - Christensen & Nielsen (2007), Bollerslev et al. (2013), Andersen & Varneskov (2021):

$$y_t = \hat{\mu} + \hat{\beta}\Delta^{\hat{d}} x_{t-1} + \hat{u}_t$$

  - **balance memory using nonlinear specifications** - Marmer (2007), Kasparis (2011), Kasparis et al. (2015), Phillips (2015)

$$y_t = \hat{\mu} + \hat{\beta} f(x_{t-1}) + \hat{u}_t$$

  - **Nonlinear filtering can rebalance both memory & tails**

Application: Predictability of Stock Returns

Table 34: IVX predictability tests; monthly data; linear fits: $f(x) = x$

$w(x)$	$x^{\langle 1 \rangle}$	$x^{\langle 0.75 \rangle}$	$x^{\langle 0.5 \rangle}$	$x^{\langle 0.25 \rangle}$	$x^{\langle 0.01 \rangle}$
BM					
Returns	0.283	0.053	0.000	0.005	0.004
Returns-Rfree	0.316	0.045	0.003	0.053	0.096
CRSP_SPvw	**3.581***	2.071	1.204	0.779	0.621
CRSP_SPvw-Rfree	**3.686***	2.014	1.032	0.522	0.287
tbl					
Returns	0.266	0.089	0	0.097	0.344
Returns-Rfree	**4.383****	**3.305***	2.014	0.818	0.127
CRSP_SPvw	0.171	0.04	0.006	0.163	0.445
CRSP_SPvw-Rfree	**4.071****	**3.047***	1.814	0.692	0.085
lty					
Returns	0.321	0.471	0.631	0.733	0.707
Returns-Rfree	**3.173***	**3.277***	**3.1***	2.569	1.807
CRSP_SPvw	0.056	0.093	0.137	0.174	0.187
CRSP_SPvw-Rfree	2.087	2.025	1.773	1.345	0.872
Inflation					
Returns	2.242	2.256	2.232	2.149	1.989
Returns-Rfree	**3.346***	**3.609****	**3.735***	**3.646***	**3.265***
CRSP_SPvw	**2.902***	**2.996***	**2.922***	2.613	2.062
CRSP_SPvw-Rfree	**4.145****	**4.534****	**4.618****	**4.248****	**3.364***
ntis					
Returns	**6.412****	**5.434****	**3.704***	1.729	0.256
Returns-Rfree	**5.975****	**4.920****	**3.195***	1.34	0.115
CRSP_SPvw	**6.520****	**5.653****	**3.971****	1.959	0.367
CRSP_SPvw-Rfree	**6.068****	**5.116****	**3.431**	1.535	0.189

Application: Predictability of Stock Returns

svar					
Returns	1.443	1.478	1.03	0.26	0.015
Returns-Rfree	0.827	0.836	0.575	0.117	0.047
CRSP_SPvw	0.035	0.061	0.12	0.3	0.704
CRSP_SPvw-Rfree	0.23	0.299	0.362	0.509	0.866
DY					
Returns	0.541	0.318	0.147	0.037	0
Returns-Rfree	0.332	0.195	0.089	0.023	0
CRSP_SPvw	0.406	0.24	0.111	0.028	0
CRSP_SPvw-Rfree	0.231	0.137	0.063	0.016	0
EP					
Returns	**3.415***	2.589	1.541	0.486	0
Returns-Rfree	1.802	1.339	0.781	0.241	0
CRSP_SPvw	1.979	1.496	0.885	0.278	0
CRSP_SPvw-Rfree	0.832	0.615	0.356	0.11	0
Dpay					
Returns	0.9	1.592	2.122	2.123	1.353
Returns-Rfree	0.264	0.679	1.063	1.13	0.675
CRSP_SPvw	0.09	0.263	0.369	0.266	0.027
CRSP_SPvw-Rfree	0.02	0.003	0.027	0.01	0.036

Thank you!