Asymptotic theory for drifting sequences of parameters in AR(p) models

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Introduction: Magdalinos and Petrova (2026)

- AR(1) process
$$x_t - \mu = \rho\left(x_{t-1} - \mu\right) + u_t$$

- Related regression models: Predictive Regression and Local Projection.
- Confidence intervals for ρ with uniform coverage over a parameter space:
$$\Theta = [-M, M], \quad \text{some } M > 0$$

- Critical regions for parameters in PR and LP models with correct asymptotic size uniformly over Θ
- Combined data-driven instrumentation procedure:
  - no discontinuities in the limit distribution
  - mixed normal limit distribution in all cases
  - standard inference (testing and CIs) based on $\mathcal{N}(0,1)$
  - correct asymptotic size (uniformly over Θ)
  - first distribution-free inference procedure in explosive case

Motivation: AR(p)

- AR(p) process

$$x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + u_t$$

- We wish to do inference on $\phi = (\phi_1, ..., \phi_p)'$

$$H_0 : H\phi_n = h, \quad \text{rank}\,(H) = q$$

- Wald statistic for testing H_0

$$W_n = \frac{1}{\hat{\sigma}_n^2} (H\tilde{\phi}_n - h)' H' \left[H \left(X' P_Z X\right)^{-1} H'\right]^{-1} (H\tilde{\phi}_n - h).$$

- $\tilde{\phi}_n$ extension of the MP(2026) IV estimator
- Z extension of the MP(2026) instrument
- Target: get $W_n \to_d \chi^2(q)$

This paper

- AR(p) process
$$x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + u_t$$
- Roots of the characteristic polynomial are the eigenvalues of the companion matrix R:
$$sp(R) = \{\rho_1, ..., \rho_p\} \subset \mathbb{C}.$$
- Discontinuity in the asymptotics of the OLS estimator of $\phi = (\phi_1, ..., \phi_p)'$ according to the properties of $sp(R)$.
- AR(p) parameter space:
$$\Theta = \{\rho \in sp(R) : \text{there exist } M, \delta > 0 \text{ s.t. } |\rho| \leq M \text{ and } sp(R) \cap (-1+\delta, 1-\delta)^c \subset \mathbb{R}\}$$
- Uniform inference on ϕ requires limit distribution theory along
$$(\rho_{1n}, ..., \rho_{pn}) \to (\rho_1, ..., \rho_p). \quad (1)$$
- Rate of convergence and limit distribution theory depends on:

1. the AR category of each $\rho_{in} \to \rho_i$ in (1)
2. the multiplicity of each ρ_{in} in (1)

Categorisation of AR regions

- $sp(R) = \{\rho_1, ..., \rho_p\}$

$$(\rho_{1n}, ..., \rho_{pn}) \to (\rho_1, ..., \rho_p).$$

- Let $c := \lim_{n\to\infty} n(|\rho_n| - 1)$ exist in $[-\infty, \infty]$.
- Then the eigenvalue ρ_n belongs to one of the following classes:

C(i) *near-stationary* if $c = -\infty$

C(ii) *LUR* if $c \in \mathbb{R}$

C(iii) *near-explosive* if $c = \infty$

Classes C(i)-C(iii) have *regular* and *oscillating* subclasses:
$C(i) = C_+(i) \cup C_-(i)$, etc

AR(p) companion matrix

- Companion matrix:

$$R = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\phi_k = (-1)^{k+1} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq p} \rho_{i_1} \ldots \rho_{i_k}$$

$$\phi_1 = \sum_{i=1}^{p} \rho_i, \quad \phi_p = (-1)^{p+1} \rho_1 \ldots \rho_p$$

- Simple Jordan form:

$$\text{geometric multiplicity} = \text{algebraic multiplicity} \tag{2}$$

- (2) implies that R is diagonalisable *iff* it has distinct eigenvalues
- In this case, adaptation of MP(2026) is relatively straightforward:

$$\text{AR}(p) \simeq p\text{-AR}(1) \text{ components}$$

- (2) implies that each group of repeated eigenvalues of R is contained in a single Jordan block which will have size equal to the algebraic multiplicity of the eigenvalue

AR(2) instrumentation

- AR(2) in companion form:

$$\begin{aligned} \underline{x}_{n,t} &= R_n \underline{x}_{n,t-1} + \underline{u}_{n,t} \qquad R_n = T_n \Lambda_n T_n^{-1} \\ T_n^{-1} \underline{x}_{n,t} &= \Lambda_n T_n^{-1} \underline{x}_{n,t-1} + T_n^{-1} \underline{u}_{n,t} \end{aligned} \tag{3}$$

- Estimated Jordan form of R_n:

$$\hat{\Lambda}_n := \begin{bmatrix} \rho_{1n} & 0 \\ 0 & \rho_{2n} \end{bmatrix} \mathbf{1}_{\Delta_n} + \begin{bmatrix} \rho_n & 1 \\ 0 & \rho_n \end{bmatrix} \mathbf{1}_{\Delta_n^c} \tag{4}$$

- We are trying to build an instrument process that follows (3) as closely as possible subject to the MP(2026) instrumentation and information on the integration order provided by (4):

$$\underline{z}_{n,t} = \tilde{\Lambda}_n \underline{z}_{n,t-1} + \tilde{\underline{u}}_{n,t}$$

$$\tilde{\Lambda}_n := \begin{bmatrix} \varphi_{1n} & 0 \\ 0 & \varphi_{2n} \end{bmatrix} \mathbf{1}_{\Delta_n} + \begin{bmatrix} \varphi_n & 1 \\ 0 & \varphi_n \end{bmatrix} \mathbf{1}_{\Delta_n^c}$$

- φ_{1n}, φ_{2n} and φ_n are artificial AR roots chosen as in MP(2026).
- Selection of $\tilde{\underline{u}}_{n,t}$ requires estimation of T_n^{-1} by OLS in (3).

Asymptotic separation of eigenvalues

- Data driven criterion for "equality" of ρ_{1n} and ρ_{2n}
- How accurately can we achieve this?
- For ρ_{1n}, ρ_{2n} in C(i)-C(ii):

$$\hat{\rho}_{1n} - \hat{\rho}_{2n} = \rho_{1n} - \rho_{2n} + O_p\left(n^{-1/2}\left(1-\rho_{1n}\right)^{1/2}\right) + O_p\left(n^{-1/2}\left(1-\rho_{2n}\right)^{1/2}\right)$$

- For LtU ρ_{1n}, ρ_{2n} in C(ii):

$$\rho_{1n} - \rho_{2n} \asymp n^{-1/2}\left(1-\rho_{1n}\right)^{1/2} + n^{-1/2}\left(1-\rho_{2n}\right)^{1/2} \asymp \frac{1}{n}$$

  impossible to distinguish!
- We need a Jordan-type decomposition that (at least) *assigns all LtU eigenvalues to the same Jordan block*.

Continuous Jordan decomposition

- AR(p) companion form:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{u}_t \quad \mathbf{u}_t = u_t \mathbf{e}_1 \tag{5}$$

  where $\mathbf{e}_1 = (1, 0, ..., 0)'$.
- There exists a triangular matrix S_n with 1 in the main diagonal and

$$\begin{aligned} (S_n)_{ij} &= h_{j-i}\left(\rho_1, ..., \rho_{p-j+1}\right), \quad ,i < j \\ h_m\left(\rho_1, ..., \rho_k\right) &= \textstyle\sum_{1 \le i_1 \le ... \le i_m \le k} \rho_{i_1} ... \rho_{i_m} \end{aligned}$$

  and

$$J_n = S_n^{-1} A S_n = \begin{bmatrix} \rho_{n,p} & 0 & 0 & 0 \\ 1 & \rho_{n,p-1} & 0 & 0 \\ & \ddots & \ddots & \\ 0 & & & 0 \\ 0 & 0 & 1 & \rho_{n,1} \end{bmatrix} \tag{6}$$

- $x \mapsto S_n^{-1} x$ on (5) gives:

$$X_t = J_n X_{t-1} + u_t \mathbf{e}_1, \qquad X_t := S_n^{-1} \mathbf{x}_t \tag{7}$$

- The vector $u_t \mathbf{e}_1$ is invariant to $x \mapsto S_n^{-1} x$

Block selection

- Decomposition:

$$X_t = diag\left(J_{1n}, ..., J_{m,n}\right) X_{t-1} + u_t \mathbf{e}_1$$

$$J_{k,n} = \begin{bmatrix} \rho_{n,k} & 0 & 0 & 0 \\ 1 & \rho_{n,k-1} & 0 & 0 \\ & \ddots & \ddots & \\ 0 & & & 0 \\ 0 & 0 & 1 & \rho_{n,1} \end{bmatrix} \tag{8}$$

- Criterion for including eigenvalues $\rho_{n,1}, ..., \rho_{n,k}$ in a block in (8):

$$\frac{1-\rho_{n,i}^2}{1-\rho_{n,j}^2} \to \gamma_{i,j} \in (0, \infty), \quad i,j \in \{1, ..., k\} \tag{9}$$

- Employ the decomposition within sub-blocks of the AR(p) process where the eigenvalues are close to each other in the sense of (9).
- Outside these blocks there is a diagonalisable component

Block selection

- Criterion for including eigenvalues $\rho_{n,1}, ..., \rho_{n,k}$ in a block in (8):

$$\frac{1-\rho_{n,i}^2}{1-\rho_{n,j}^2} \to \gamma_{i,j} \in (0,\infty), \quad i,j \in \{1,...,k\}$$

- $\gamma_{ij} = 1$ approximates the usual Jordan decomposition.
- (9) gives us more flexibility: it allows us to put all local to unity roots in one block
- Treat $\rho_{n,i}$ and $\rho_{n,j}$ as different iff $\gamma_{ij} = 0$ or $\gamma_{ij} = \infty$ which will happen iff one of the following happens:

1. $\rho_{n,i}$ and $\rho_{n,j}$ belong to different AR classes $\{$C(i), C(ii), C(iii)$\}$
2. $\rho_{n,i}$ and $\rho_{n,j}$ are both in C(i) or in C(iii) and $\rho_{n,i} \to 1$ whereas $\rho_{n,j} \not\to 1$
3. $\rho_{n,i}$ and $\rho_{n,j}$ are both in C(i) or in C(iii) and $\rho_{n,i} \to 1$ strictly faster/slower that $\rho_{n,j} \to 1$

- (9) does not distinguish between different purely stationary or between purely explosive roots $\rho_{n,i}$ and $\rho_{n,j}$

OLS asymptotics within blocks

- Decomposition:

$$X_t = diag\left(J_{1n}, ..., J_{m,n}\right) X_{t-1} + u_t \mathbf{e}_1 \tag{10}$$

$$J_{k,n} = \begin{bmatrix} \rho_{n,k} & 0 & 0 & 0 \\ 1 & \rho_{n,k-1} & 0 & 0 \\ & \ddots & \ddots & \\ 0 & & & 0 \\ 0 & 0 & 1 & \rho_{n,1} \end{bmatrix}$$

- OLS asymptotics within each block in (10) when

$$\gamma_{i,j} = \lim_{n\to\infty} \frac{1-\rho_{n,i}^2}{1-\rho_{n,j}^2} \in (0,\infty), \tag{11}$$

$$\Gamma_{j,l} = \lim_{n\to\infty} \frac{1-\rho_{n,j}^2}{1-\rho_{n,j}\rho_{n,l}} \in (0,\infty) \tag{12}$$

- Each block of eigenvalues necessarily belongs to one AR class from {C(i), C(ii), C(iii)}
- OLS asymptotic theory generalises work of Chan and Wei (1988) and Jaganathan (1988)

OLS asymptotics: near-stationary blocks

- $M_n = \sum_{j=0}^{\infty} J_n^j \mathbf{e}_1 \mathbf{e}_1' \left(J_n^j\right)'$ satisfies

$$\det(M_n) = \frac{1}{\prod_{i=1}^{p}\prod_{j=1}^{p}\left(1-\rho_{n,i}\rho_{n,j}\right)}$$

- Normalisation matrix:
$D_n = diag\left(\left(1-\rho_{n,p}^2\right)^{-1/2}, \left(1-\rho_{n,p-1}^2\right)^{-3/2}, ..., \left(1-\rho_{n,1}^2\right)^{-(p-1/2)}\right)$

Theorem

If all eigenvalues $\rho_{n,i}$ in J_n are in C(i) and satisfy (9), then

$$\frac{1}{n} D_n^{-1} \sum_{t=1}^{n} X_{t-1} X_{t-1}' D_n^{-1} \to_p \sigma^2 V$$

where $V = \lim_{n\to\infty} D_n^{-1} M_n D_n^{-1}$ satisfies

$$\det V = \gamma_{p,1}\gamma_{p-1,1}^3 \cdots \gamma_{2,1}^{2p-3} \prod_{i=1}^{p} \gamma_{i1}^{-p} \Gamma_{i1} \ldots \Gamma_{ip} > 0.$$

OLS asymptotics: near-explosive blocks

- Normalisation matrix:
$$D_n = diag\left(\left(\rho_{n,p}^2 - 1\right)^{-1/2}, \left(\rho_{n,p-1}^2 - 1\right)^{-3/2}, ..., \left(\rho_{n,1}^2 - 1\right)^{-(p-1/2)}\right)$$

Theorem

If all eigenvalues $\rho_{n,i}$ in J_n are in C(iii) and satisfy (9), then

$$W_n := \left(\rho_{n,p}^2 - 1\right)^{1/2} D_n^{-1} \sum_{k=1}^{n} u_k J_n^{-k} D_n \to_d W \tag{13}$$

where the matrix W satisfies

$$\det W = \left(\gamma_{p,p-1} ... \gamma_{p,1}\right)^{1/2} \zeta_1 ... \zeta_p \neq 0 \quad a.s.$$

where $\mathbb{E}\zeta_i = 0$, $\mathbb{E}\zeta_i^2 = \sigma^2$ and ζ_i are Gaussian when $\rho_i = 1$. Moreover

$$\left(\rho_{n,p}^2 - 1\right) D_n^{-1} J_n^{-n} \sum_{t=1}^{n} X_{t-1} X_{t-1}' \left(J_n^{-n}\right)' D_n^{-1} \to_d W\,(-V)\,W > 0 \quad a.s.$$

OLS asymptotics: LtU blocks

- Normalisation matrix: $D_n = diag\left(n^{1/2}, n^{3/2}, ..., n^{p-1/2}\right)$
- $n(\rho_{ni} - 1) \to c_i \in \mathbb{R}, \quad c = (c_1, ..., c_p)'$,

$$X_{\lfloor nt \rfloor} = \sum_{k=1}^{\lfloor nt \rfloor} J_n^{\lfloor nt \rfloor - k} u_k \mathbf{e}_1 \quad \text{and} \quad J_{c_i}(t) = \int_0^t e^{c_i(t-s)} dB(s).$$

Theorem

If all eigenvalues $\rho_{n,i}$ in J_n are in C(ii) and satisfy (9), then, the following FCLT applies on $D[0,1]^p$:

$$D_n^{-1} X_{\lfloor nt \rfloor} \Rightarrow U_c(t)$$

where $U_c(t)$ is a zero mean Gaussian process on $D[0,1]^p$ with i-th element

$$U_{c,i}(t) = \int_0^t e^{c_p \alpha_1} ... \int_0^t e^{c_{p-i+2} \alpha_{i-1}} J_{c_{p-i+1}}(t - \alpha_{i-1} - ...\alpha_1) d\alpha_{i-1} ... d\alpha_1.$$

Moreover, $n^{-1} D_n^{-1} \sum_{t=1}^n X_{t-1} X_{t-1}' D_n^{-1} \to_d \int_0^1 U_c(t) U_c(t)' dt.$

Conclusion

- We establish OLS limit theory using a Jordan-like decomposition that allows:
  - instrumentation that estimates the Jordan form
  - asymptotically positive definite sample moment matrices
  - implies asymptotically positive definite normalised $Z'Z$ instrument matrices
- Still to do for uniform inference in AR(p) processes:
  - refine the eigenvalue separation criterion
  - scale up the analysis of

$$D_n^{-1} \frac{1}{n} \sum_{t=1}^{n} X_{t-1} Z_{t-1}' D\left(\varphi_n\right)^{-1} \rightarrow_d \Psi, \quad \det\left(\Psi\right) \neq 0 \ a.s.$$

  from AR(2) to AR(p)