The Risk-free Rate and the Risk-adjusted Growth Rate

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Motivation

- A conundrum
  - The government's intertemporal budget constraint: The value of debt equals the present value of primary surpluses: Debt = Present value of future primary surpluses
  - But: primary surpluses are negative as far as the eye can see...
  - Yet... the value of debt is positive...
- Two possible resolutions:
  - So, either the market expects positive surpluses at some point in the future, or
  - Debt $\neq$ Present value of future primary surpluses

Motivation

- A theoretical issue:
  - Under what condition can the value of an asset differ from the present value of its dividends?
  - In a deterministic economy: $r < g$. The dynamics of the debt-to-GDP ratio:

$$db_t = (r - g)b_t dt + \frac{D_t}{Y_t} dt$$

  - If there exists a steady state with $r - g < 0$, then the steady state value of b_t is

$$b = \frac{D}{Y} \frac{1}{g - r}$$

  - So, it is possible to have a perennial deficit, and yet positive value of debt.

Motivation

- An implication of the deterministic model. If $r - g < 0$, then the present value of the aggregate endowment must be infinite.
- Intuition: The Gordon-Growth formula:

$$PV(Y) = \frac{Y}{r - g} \rightarrow \text{becomes ill-defined when } r-g \text{ is negative.}$$

- A popular argument: If there is a non-depreciating asset (such as land), and there is balanced growth, there can be no bubbles in a deterministic economy.
  - The value of land is finite.
  - In steady state, the rents of land are a constant fraction of GDP.
  - .... So if the value of land is finite, the value of the aggregate endowment must be finite.
  - and there can be no bubbles.

Motivation

- Does this argument extend to a stochastic economy?
- Two things change in a stochastic setting:
  1. The discount factor is stochastic.
  2. The rents of land are merely cointegrated with the aggregate endowment.
- Santos Woodford (1995): finite present value of the aggregate endowment is finite $\rightarrow$ no bubbles
- But ... the finite value of a non-depreciating asset does not imply the finite value of the aggregate endowment.
  - .... even in the presence of cointegration.

A conundrum

- Dividend strips of two cointegrated cash flows
  - Take two cash flow processes X_t and C_t that are co-integrated and assume $PV(X_t) < \infty$.
  - Consider the prices of dividend strips to X_T and C_T for some large T.
  - It would appear that the yields of long-term dividend strips to C_T and X_T are the same.
  - So, if the dividend strips of X_t are summable, so should the dividend strips to C_t.

- While intuitively appealing, the above line of reasoning is entirely false.
  - The term structures of two cointegrated claims can be completely unrelated both in the short and in the long run.
  - Implications for the "Debt valuation puzzle".

Outlook

- Explain why we can't use a high-level argument (such as the finiteness of the value of land) to preclude the possibility of bubbles.
- Essentially explain why two cointegrated cash flows can have very different asymptotic dividend-strip properties.
- Report on a historical experiment, where growth-continget bonds were traded.
  - These bonds allow us to gauge whether (at least for interemediate horizons) the risk free rate exceeds the risk-adjusted growth rate.
- Main finding: The risk-free rate was below the risk-adjusted growth rate.

Cointegrated payout process

- Consider the process

$$\frac{dC_t}{C_t} = gdt + \sigma_C dB_t^C,$$

- A fund can invest in N assets

$$\underbrace{dR_t}_{N\times 1} = \underbrace{\mu}_{N\times 1} dt + \underbrace{\sigma}_{N\times M}\underbrace{dB_t}_{M\times 1}, \text{ where } dB_t \equiv \begin{bmatrix} dB_t^C \\ \overrightarrow{dB}_t \end{bmatrix},$$

- $N \leq M$ and σ has rank N. Market is either complete $(N = M)$ or incomplete $(N < M.)$

- Consider a fund that can make distributions equal to X_t

$$\frac{dW_t}{W_t} = \left(r + w'\mu^e - \frac{X_t}{W_t}\right) dt + w'\sigma dB_t$$

Cointegrated payout process

- Let $z_t \equiv \log W_t - \log C_t$ denote the fund-value-to-cash-flow ratio.
- Suppose that for some constants α, $\beta > 0$, the fund makes distributions according to the following rule:

$$\frac{X_t}{W_t} \equiv \max(\alpha + \beta z_t, \varepsilon), \text{ where } \beta > 0 \text{ and } \varepsilon > 0.$$

- Define A, the difference between the logarithmic rate of rate of return on the portfolio w and the logarithmic growth rate of C_t:

$$A = A(w) \equiv r - g + w'\mu^e - \frac{w'\sigma\sigma'w}{2} + \frac{\sigma_C^2}{2}.$$

Cointegrated payout process

Proposition

Consider any w that is not the replicating portfolio. If $A(w) > \varepsilon$, then z_t is stationary.

- The (log) distributions made by the fund, X_t are co-integrated with the (log) cash-flow process C_t, since

$$\log X_t - \log C_t = \log \frac{X_t}{W_t} - \log \left(\frac{C_t}{W_t} \right) = \log(\max(\alpha + \beta z_t, \varepsilon)) + z_t$$

An illustration of Proposition 1

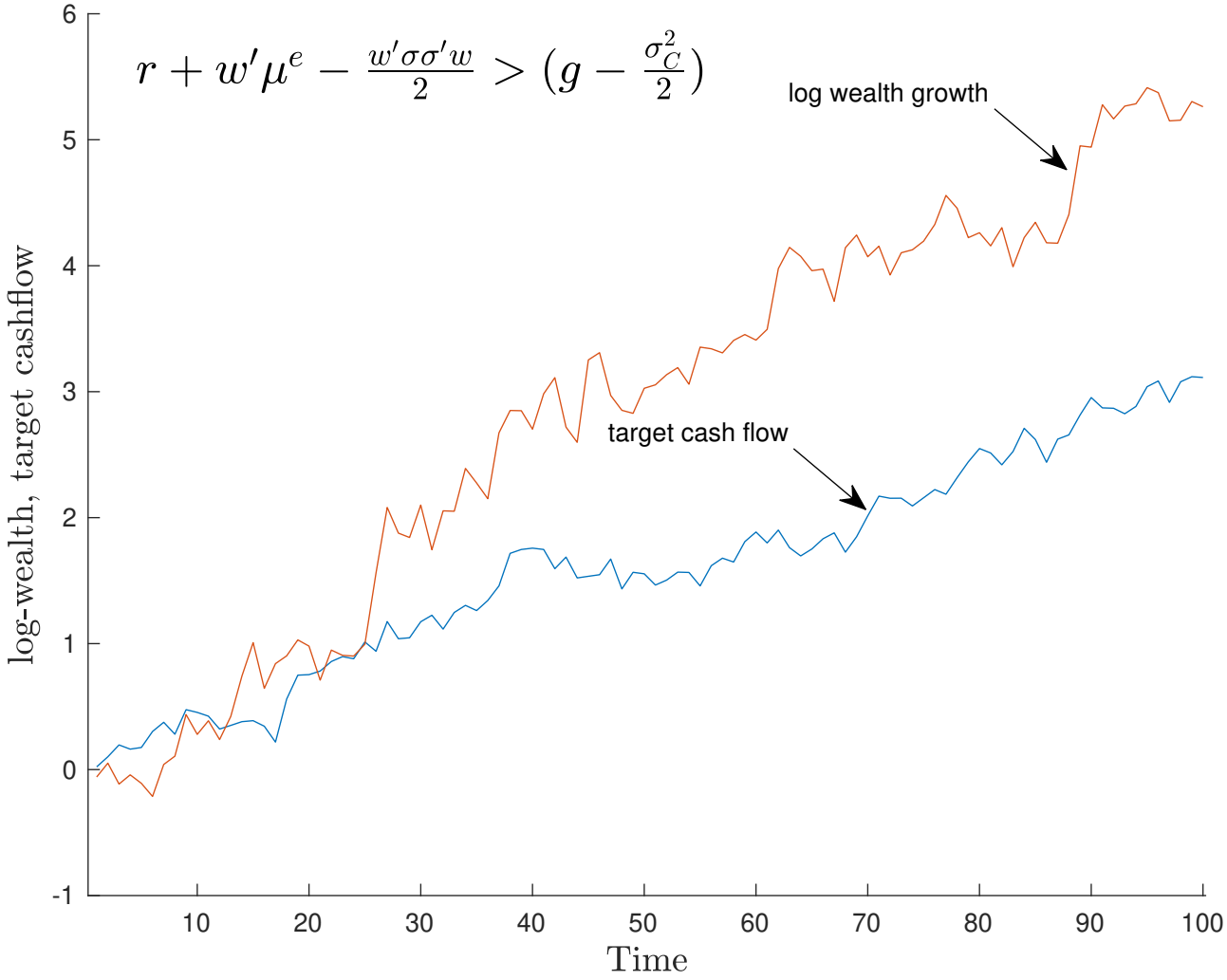

Figure: An illustration of the assumptions of Proposition 1

An illustration of Proposition 2

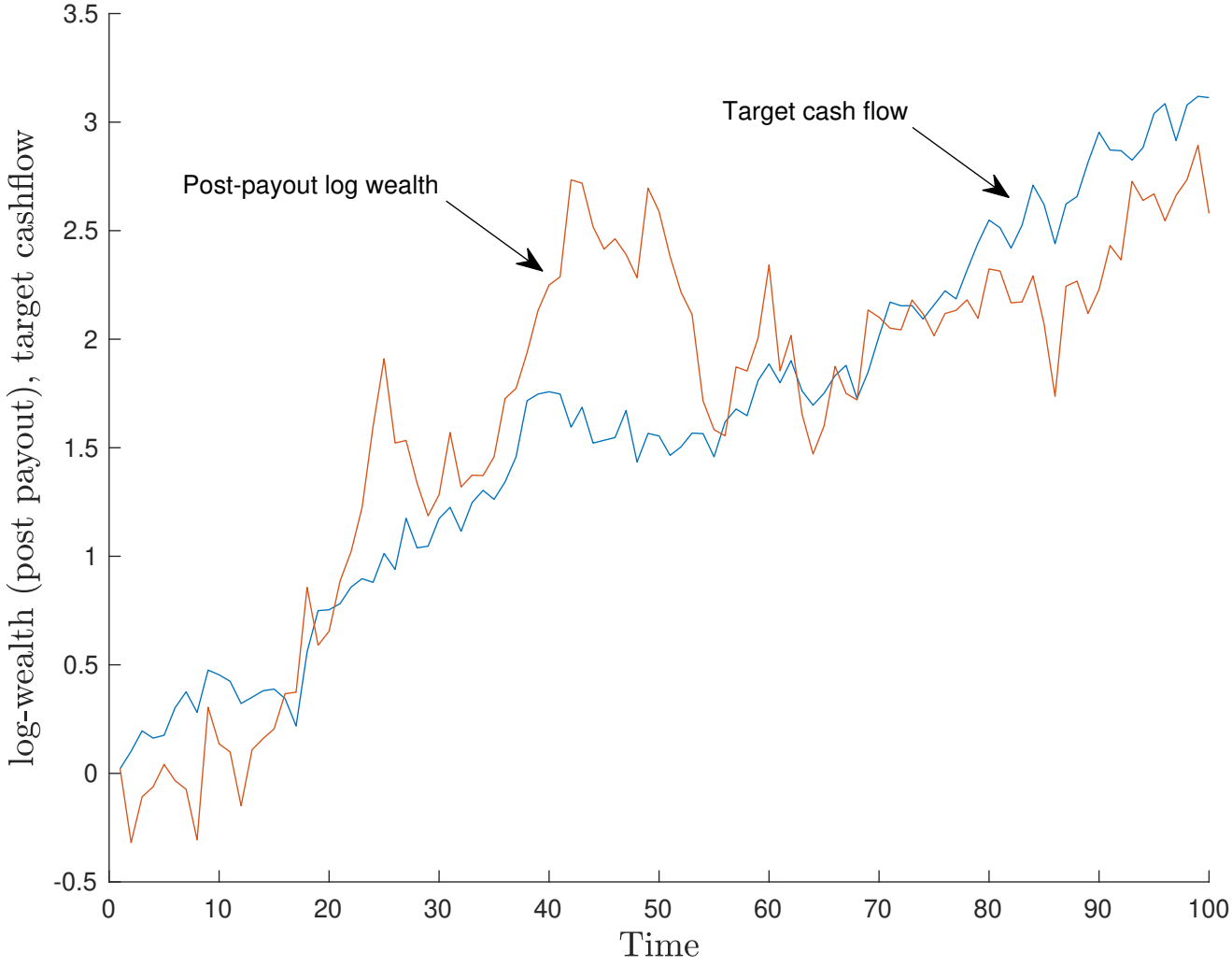

Figure: Post-payout log-wealth growth

Long Dated Strips

- As long as there exists a portfolio w such that $A(w) > 0$, it is always possible to find a payout strategy, X_t, that is co-integrated with C_t.
- Even though the cash-flow processes $\log X_t$ and $\log C_t$ are co-integrated,
  - the value of long-dated divided "strips" to the cash flows X_T and C_T at time T can diverge as $T \to \infty$
- More generally, even if the value of both strips converges to zero as $T \to \infty$, the yields of these strips will in general not converge to the same number as $T \to \infty$.

Long Dated Strips

- To remove any ambiguity about the notion of "present value" let's focus on $N = M$, so that the market is complete.
- Define the unique stochastic discount factor as

$$\frac{dm_t}{m_t} = -r dt - \underbrace{\kappa}_{N \times 1} \begin{bmatrix} dB_t^c \\ d\vec{B}_t \end{bmatrix},$$

- r is the prevailing interest rate and the vector $\kappa \equiv \sigma^{-1}\mu^e$ is the vector of the "market prices of risk."

Long Dated Strips

- The price $P_t^{(X_T)}$ of a dividend strip to the cash flow X_T, divided by the current level of X_t is defined as

$$\frac{P_t^{(X_T)}}{X_t} \equiv E_t\left(\frac{m_T}{m_t}\frac{X_T}{X_t}\right) = E_t^Q\left(e^{-r(T-t)}\frac{X_T}{X_t}\right)$$

where Q is the "risk-neutral" measure associated with the SDF

- Similarly, the value of a long-dated dividend-strip to the cash flow C_t is given by

$$\frac{P_t^{(C_T)}}{C_t} = E_t^Q\left(e^{-r(T-t)}\frac{C_T}{C_t}\right) = e^{-\left(r-g+\kappa^{(1)}\sigma_C\right)(T-t)}$$

where $\kappa^{(1)}$ is the price of risk associated with B_t^C).

  - The quantity $g - \kappa^{(1)}\sigma_C$ can be thought of as the "risk-adjusted" growth rate of C_t.
  - In particular, if $r < g + \kappa^{(1)}\sigma_C$, then $\lim_{T\to\infty}\frac{P^{(C_T)}}{C_t} = \infty$.

Long Dated Strips

Proposition

There exists (an open set of) parameters such that (a) $r - g + \kappa^{(1)}\sigma_C < 0$ *and (b) the conditions of the earlier Proposition* ($A(w) > 0$) *both hold. Therefore,* $\log \frac{X_T}{C_T}$ *is stationary, and yet* $\lim_{T\to\infty} P_t^{(C_T)} = \infty$, *while* $\lim_{T\to\infty} P_t^{(X_T)} = 0$.

Long Dated Strips - Intuition

- Define $\boxed{f(z_t) \equiv \frac{X_t}{C_t}}$

$$\underbrace{\frac{1}{T-t}\log\left(\frac{P_t^{(X_T)}}{X_t}\right)}_{\text{Yield on } X} = \frac{1}{T-t}\log E_t^Q\left(e^{-r(T-t)}\frac{X_T}{X_t}\right)$$

Long Dated Strips - Intuition

- Define $\boxed{f(z_t) \equiv \frac{X_t}{C_t}}$

$$\underbrace{\frac{1}{T-t}\log\left(\frac{P_t^{(X_T)}}{X_t}\right)}_{\text{Yield on } X} = \frac{1}{T-t}\log E_t^Q\left(e^{-r(T-t)}\frac{C_T f\left(z_T\right)}{C_t f\left(z_t\right)}\right)$$

Long Dated Strips - Intuition

- Define $\boxed{f(z_t) \equiv \frac{X_t}{C_t}}$

$$\underbrace{\frac{1}{T-t}\log\left(\frac{P_t^{(X_T)}}{X_t}\right)}_{\text{Yield on } X} = \frac{1}{T-t}\log\left[e^{-r(T-t)}\frac{E_t^Q\left(C_T\right)}{C_t} \times E_t^Q\left(\frac{C_T}{E_t^Q\left(C_T\right)}\frac{f\left(z_T\right)}{f\left(z_t\right)}\right)\right.$$

Long Dated Strips - Intuition

- Define $\boxed{f(z_t) \equiv \frac{X_t}{C_t}}$

$$\underbrace{\frac{1}{T-t}\log\left(\frac{P_t^{(X_T)}}{X_t}\right)}_{\text{Yield on } X} = \underbrace{\frac{1}{T-t}\log\frac{P_t^{(C_T)}}{C_t}}_{\text{Yield on } C} + \frac{1}{T-t}\log E_t^Q\left(\frac{C_T}{E_t^Q(C_T)}\frac{f(z_T)}{f(z_t)}\right)$$

Long Dated Strips - Intuition

- Denote

$$\mathcal{R} \equiv \lim_{T\to\infty} \frac{1}{T-t} \log E_t^Q \left(\frac{C_T}{E_t^Q\left(C_T\right)} \frac{f(z_T)}{f\left(z_t\right)} \right) = \lim_{T-t\to\infty} \frac{1}{T-t} \log E_t^{\widetilde{Q}} \left(\frac{f(z_T)}{f\left(z_t\right)} \right),$$

- where we used the change of measure $\frac{d\widetilde{Q}}{dQ} = \frac{C_T}{E_t^Q(C_T)}$.

- If we knew that $f(z_t)$ is stationary under $\widetilde{Q}$, then $\mathcal{R} = 0$.

- However, it is possible that the cointegrating vector z_t is stationary under P (the data generating process), and yet non-stationary under $\widetilde{Q}$.

Long Dated Strips - Intuition

Lemma

Under the measure $\widetilde{Q}$, the dynamics of z_t are

$$dz_t = \left(\widetilde{A}(w) - \max(\alpha + \beta z_t, \varepsilon)\right) dt + \phi' dB_t^{\widetilde{Q}},$$

where $\phi' = \left\{w'\sigma - \left[\sigma_C, 0_{1\times(N-1)}\right]'\right\}$ *and* $\widetilde{A}(w)$ *is given by*

$$\widetilde{A}(w) = r - \left(g - \kappa^{(1)}\sigma_C\right) - \frac{\phi'\phi}{2}.$$

- If $r - \left(g - \kappa^{(1)}\sigma_C\right) < 0$, then $\widetilde{A}(w) < 0$.
  - Note that only the *first element*, $\kappa^{(1)}$, of the vector of Sharpe ratios matters for the sign of $r - \left(g - \kappa^{(1)}\sigma_C\right)$, and by implication $\widetilde{A}(w)$.
- By contrast $A(w)$ depends on the entire vector of Sharpe ratios κ. So it is very well possible that $A(w) > 0 > \widetilde{A}(w)$.

Long Dated Strips - Not just an asymptotic phenomenon

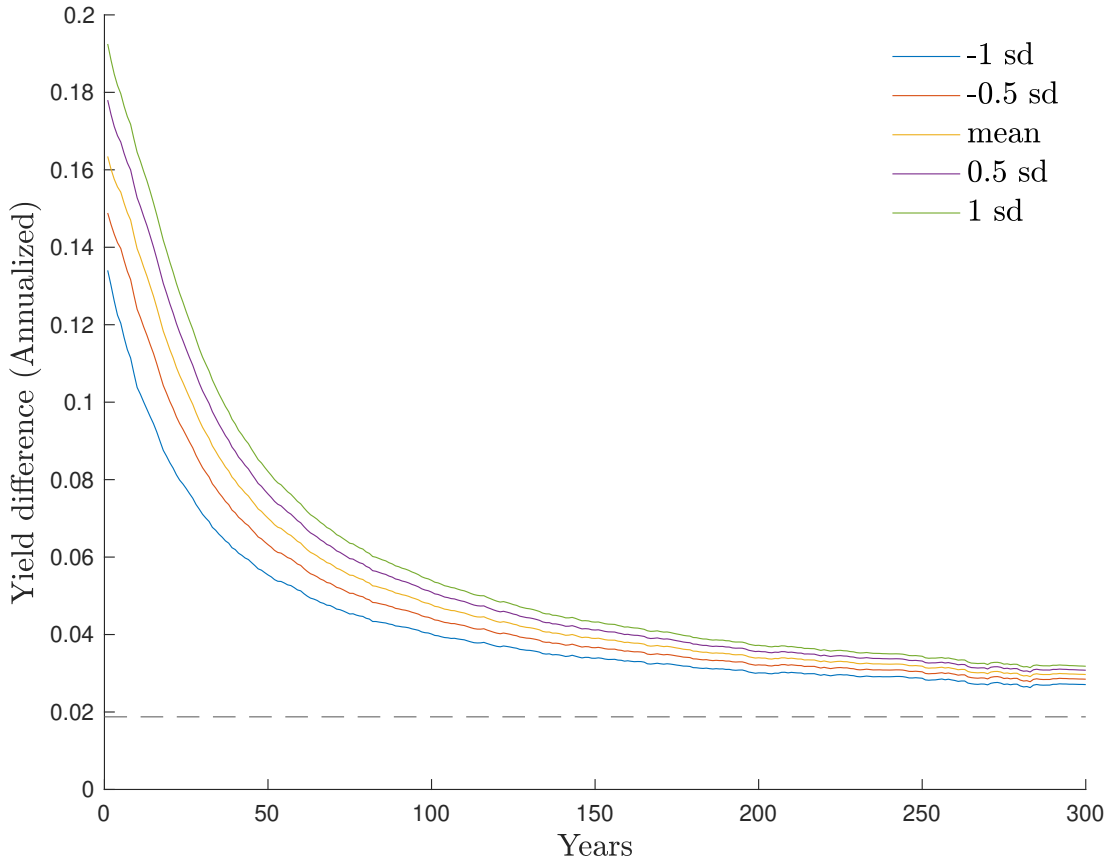

Figure: Difference in the yields of dividend strips to X_T and C_T. The different lines correspond to initializing z_t at the stationary mean of z (under the statistical measure, P) plus / minus $0, 0.5, 1$ stationary standard deviations. The value of $\widetilde{A}(w)$ is -0.02.

Outlook

- Taking stock: By observing that the value of land is finite, and the rents of land are cointegrated with GDP, we cannot conclude a lot about the presence or absence of bubbles.
- Short of that, we have to basically find out some way to determine if the risk-adjusted growth rate of the endowment exceeds the interest rate.
- It would be great if we could find some claims whose cash flows are indexed to aggregate growth. Because the prices of this claim would give us directly

$$\frac{P_t^{(C_T)}}{C_t} = E_t^Q\left(e^{-r(T-t)}\frac{C_T}{C_t}\right) = e^{-\left(r-g+\kappa^{(1)}\sigma_C\right)(T-t)}$$

The French Experience with Growth-Contingent Bonds

- To infer the risk premium for "pure aggregate risk" one would need data on growth-contingent bonds, or GDP futures.
- Very few examples of such contracts. Typically after sovereign defaults.
- The only case of a major industrialized country that introduced such bonds: France in the mid-1950s ("Bons d'Equipement Industriel et Agricole")
- La Monde (May 24, 1956) press release
  - denominations of 10,000 ,100,000, and 1,000,000 francs
  - 15-year maturity.
  - Redemption would be at 105% of par.
  - Coupon would be at a 5% minimum; in addition, the coupon increased by 0.05% for each point that the annual index of industrial production exceeds its level in 1955 (base 120 in 1955.)
  - "interest and the increase in interest are exempt from tax on securities, the redemption premium benefits from the same exemption and is not subject to the progressive surtax on personal income."

Data

- Traded on the French stock exchange and are available on the "Database for Financial History" (dfih.fr).
- The database contains 875 price observations (typically on a biweekly freqeuncy from August 1956 to April 1971) across all three denominations.
- The database indicates that the data corresponds to actual transactions, because the entry "transaction #" takes numbers larger than 1 on several days.
- Additional data series:
  1. Annual data on French real GDP 1950-1971 from FRED.
  2. Monthly data on "Production of Total Industry in France", available from 1960-1971
  3. Monthly data on 10-year yields available from 1960-1971. Also Tableu 3 from Chauveau(1975) for the years 1953-1972.
- Limitations:
  1. Single issue
  2. Not clear which index is used as reference and of what date

Industrial Production and GDP

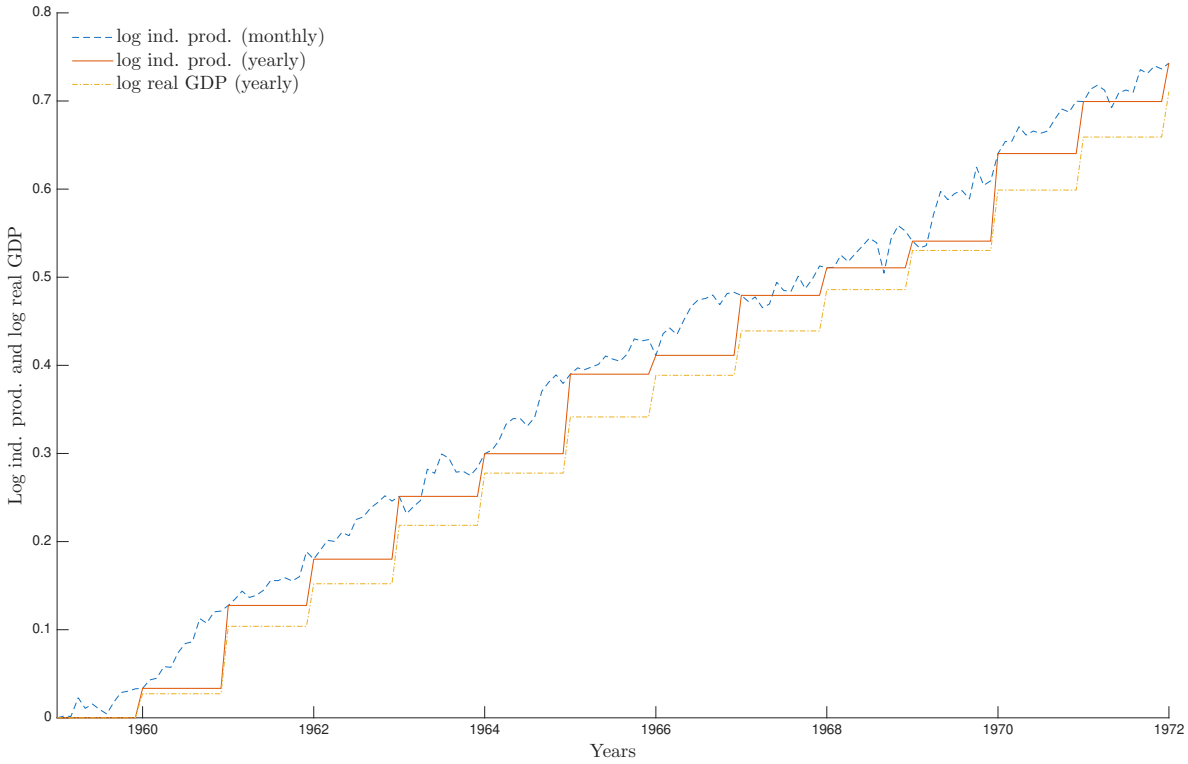

Figure: Log real GDP (yearly) and log industrial production (monthly and yearly). Base year=1959. The monthly observations in March 1963 and May-June 1968, which represent a transient drop in the monthly industrial production series, are omitted from the figure.

Empirical analysis

- Assume that the dynamics of GDP follow the process

$$\frac{dY_t}{Y_t} = g^Q dt + \sigma dB_t^Q$$

- Price of the bond $\widehat{P}(Y_t)$ at time t is

$$\begin{aligned}\widehat{P}(Y_t) &= 100 \times \sum_{T_i}^{T} e^{-r_t(T_i - t)} \left[0.05 + 0.0005 \times E_t^Q \left(Y_t - Y_{1955}\right)^+\right] + 105 e^{-r_t(T-t)} \\ &= 100 \times \sum_{T_i}^{T} \left[e^{-r_t(T_i - t)} 0.05 + 0.0005 \times C^{BS}\left(Y_t, Y_{1955}, T_i - t, r_t, \sigma, r_t - g^Q\right)\right] \\ &\quad + 105 e^{-r_t(T-t)}.\end{aligned}$$

- This equation allows inference of Y_t by solving the equation $P_t = P(Y_t)$, where P_t is the observed price of the security at time t.
- This leads to the implicit value of industrial production $\widetilde{Y}_t(P_t; g^Q, \sigma) = \widehat{P}^{-1}\left(P_t; g^Q, \sigma\right)$.

Empirical analysis

- Define the residual $u_{t_i} \equiv \log Y_{t_i} - \log \widetilde{Y}_{t_i}(P_{t_i}; g^Q, \widehat{\sigma}) - \alpha$ and estimate the risk-neutral growth parameter, g^Q, to minimize the expression

$$\widehat{g}^Q = \arg\min_{g^Q, \alpha} \sum_{i=1}^{N} u_{t_i}^2.$$

- Derive the standard errors analytically by using
  1. Implicit function theorem
  2. The Black Scholes "Greeks"

Empirical analysis: Model and Data fit

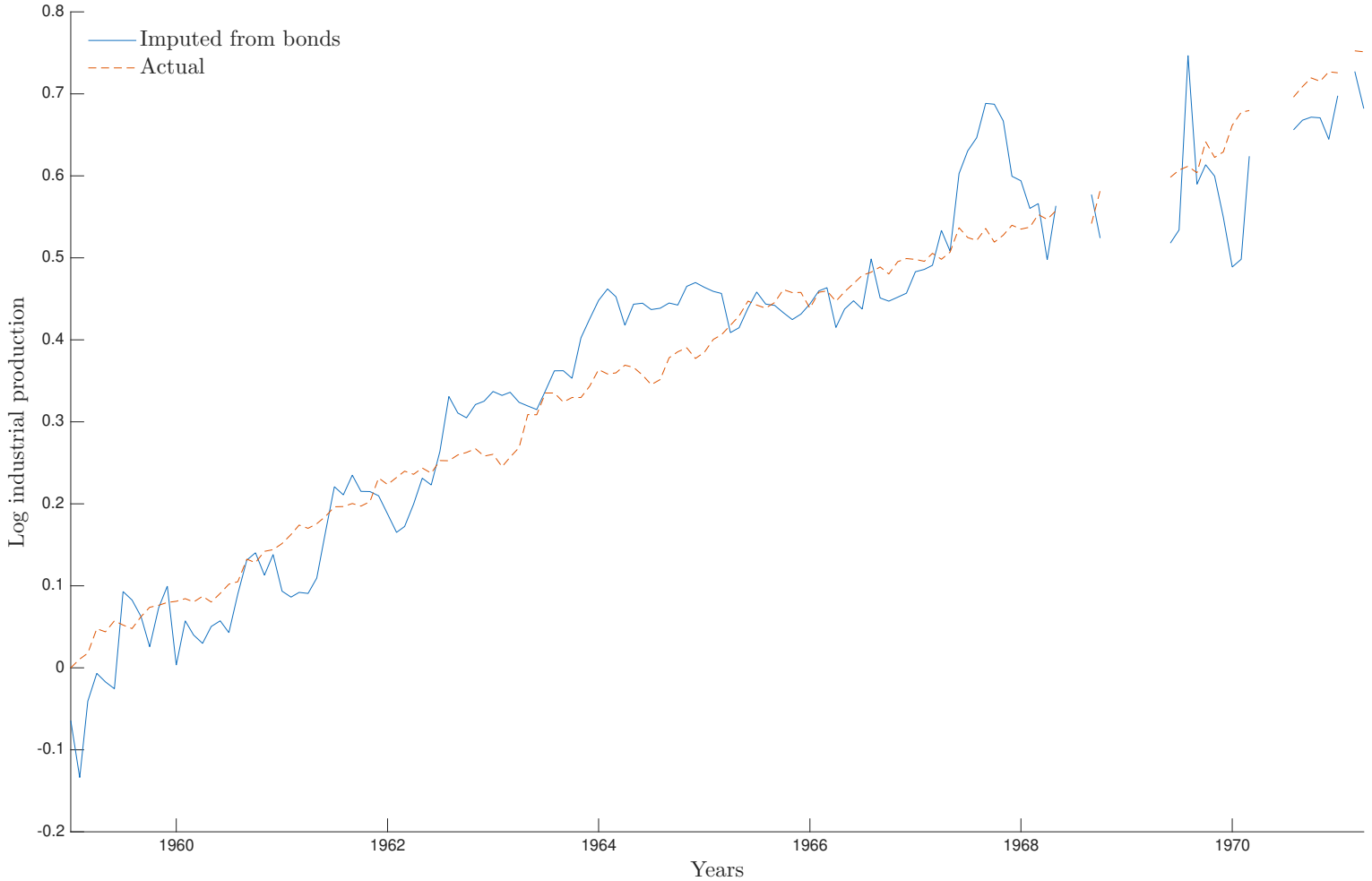

Figure: Log-Industrial production index inferred from bond prices (solid line), and the index of log industrial production (dotted line).

Main result

	One sample	Two subsamples	
	g^Q	$g^Q_{\text{first half}}$	$g^Q_{\text{second half}}$
σ =0.042	0.0326	0.0331	0.0344
	(0.0074)	(0.0091)	(0.0165)
σ =0.020	0.0329	0.0335	0.0352
	(0.0073)	(0.0089)	(0.0163)
σ =0.030	0.0328	0.0334	0.0349
	(0.0074)	(0.0090)	(0.0164)
σ =0.050	0.0324	0.0329	0.0341
	(0.0075)	(0.0091)	(0.0166)
σ =0.060	0.0322	0.0326	0.0337
	(0.0076)	(0.0092)	(0.0167)

Table: Estimates of g^Q for different subsamples and different volatility assumptions.

Empirical analysis: Model and Data fit

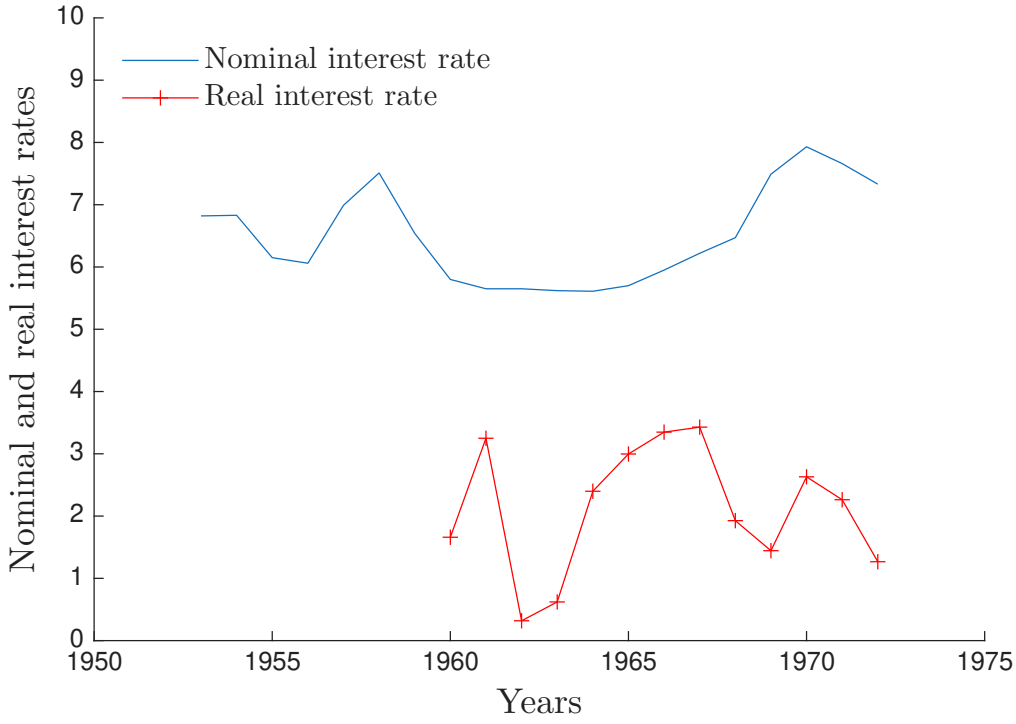

Figure: Nominal Interest rates (from long-term bonds), and the real interest rate computed as the difference between the nominal interest rate and inflation. (Data on inflation are available from 1960 onward.)

Conclusion

- There is no "quick and simple" argument to preclude the possibility of bubbles
  - Existence of finitely-valued, non-depreciating assets $\neq$ absence of bubbles .
  - Reason: Cointegration of two cash flows $\neq$ equality of long-term yields on strips
- Historical experience of France
  - Suggests $g^{\mathcal{Q}} > r$.
  - The gap $g^{\mathcal{Q}} - r$ provides a reading of the deficit that can be financed while ensuring a deterministic path for the debt-to-GDP ratio.